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Localized tachyons in $\mathbb{C}^3/\mathbb{Z}_N$

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Abstract: We study the condensation of localized closed string tachyons in $\mathbb{C}^3/\mathbb{Z}_N$ nonsupersymmetric noncompact orbifold singularities via renormalization group flows that preserve supersymmetry in the worldsheet conformal field theory and their interrelations with the toric geometry of these orbifolds. We show that for worldsheet supersymmetric tachyons, the endpoint of tachyon condensation generically includes "geometric" terminal singularities (orbifolds that do not have any marginal or relevant Kähler blowup modes) as well as singularities in codimension two. Some of the various possible distinct geometric resolutions are related by flip transitions. For type-II theories, we show that the residual singularities that arise under tachyon condensation in various classes of type-II theories also admit a type-II GSO projection. We further show that type-II orbifolds entirely devoid of marginal or relevant blowup modes (Kähler or otherwise) cannot exist, which thus implies that the endpoints of tachyon condensation in type-II theories are always smooth spaces.

Keywords: Renormalization Group, Tachyon Condensation.

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Contents

1. Introduction

Following the seminal insights of Sen [1], the last few years have seen the emergence of an understanding of tachyon dynamics in string theory. This development is important both as a foothold on time evolution in string theory, which has until now been difficult to study, and because it allows us to study some properties of string vacua by considering them as the endpoint of tachyon condensation starting from simpler solutions. Open string tachyons, being localized to D-brane worldvolumes, have relatively controlled descriptions, obtained by taking limits such that the open string dynamics decouples from the more complicated dynamics of the bulk closed-string theory. In particular, this approach eliminates the complications of gravitational backreaction.

Closed string tachyons can also exist in configurations in which they decouple from most of the string modes, and in particular from gravity. Consider strings propagating on a space with singularities. In an appropriate large radius limit, string modes localized at the singularity decouple from the bulk theory (a clear general review of this approach is [2]). In particular, one can engineer models in which no tachyons propagate in the bulk while localized tachyons exist at the singular locus, and study the condensation of these localized tachyons. A particularly simple class of singularities are orbifolds (quotient singularities), analyzed in [3, 4, 5] (considerable work has been done on closed string tachyon condensation: a recent review with a relatively complete list of references is [6]). In the decoupling limit, the local dynamics near an orbifold point can be well approximated by the dynamics of string propagation on the tangent cone $\mathbb{C}^n/\Gamma$, i.e. as an orbifold of a free conformal field theory, and is hence amenable to explicit calculations.

In fact, what we study is not directly the time evolution of the system. Instead, we note that a static tachyon condensate of course breaks the conformal symmetry. It corresponds to a relevant deformation of the worldsheet theory, and we can study the worldsheet renormalization group flows generated by such deformations. In particular, we can consider the endpoints of such flows: as conformal field theories these correspond to string vacua, and the RG flow thus realizes a path from one vacuum to another, more stable one. While the details of the path are almost certain to differ from the dynamical time evolution, the endpoints of the flow agree in known examples with the asymptotic future of the time-dependent solutions.

The free field description of the orbifold theory allows a simple and direct calculation of the spectrum. Localized states arise in twisted sectors, and in appropriate models all tachyons are twisted states. Unfortunately, this means that in general the free field description does not lead to simple descriptions of the deformed theories after condensation. If the worldsheet theory with which we start has an $\mathcal{N} = (2, 2)$ superconformal symmetry, there is a class of deformations for which powerful constraints can be used to control the RG flow [5]. These are chiral primary deformations.[1] While breaking the conformal symmetry they preserve the full supersymmetry, simplifying the description of the deformed models. This simplification is related to the existence of a twisted topological version of the theory, retaining only the chiral primary fields.

These decays are also distinguished in that they can be given a clear geometric interpretation. They correspond to Kähler deformations (partially) resolving the orbifold singularity, in the sense that the (non-conformal) supersymmetric field theory obtained by deforming the action is a nonlinear sigma model on the Kähler space obtained by deforming the quotient. The geometric interpretation provides a global setting for the renormalization group flows, and our understanding of Kähler deformation can be used to study global aspects of the flow. The description of the resolved space as a toric variety has proved particularly useful in these investigations [5, 7] and will again be useful here. It turns out that we can associate operators directly to particular toric deformations. Taking the extreme limit of these - when the sizes of all exceptional sets are taken to infinity - we find in general several abelian quotient singularities in an otherwise smooth space. In general, there

[1] More precisely, the action is deformed by adding the integral of the top component of a superfield whose lowest component is a chiral primary field.

will still be localized tachyons associated to these, and their condensation will continue the process of resolving the singularity. The geometric description, and in particular the description of the resolved orbifold as a toric variety, was used in [5] to find the endpoints of the decay for cyclic quotient singularities ($\Gamma = \mathbb{Z}_n$) in complex codimension one and two. In this paper, we extend this to cyclic quotient singularities in codimension three.

Some new features of this case are noteworthy. A sequence of resolutions in the case of codimension two quotients ends either with a smooth space or with quotients by discrete groups contained in SU(2). We will call the latter supersymmetric quotients, because a type-II string compactification on such a space leads to spacetime supersymmetry. A supersymmetric quotient has no relevant operators, but we can still move out along marginal directions to resolve the singularity completely. On the other hand, tachyons induce RG flows: under such a flow to the IR induced by a given tachyon, the anomalous dimensions of the remaining chiral operators in general increase, as we show using toric methods. In particular in codimension three, some operators relevant in the UV become irrelevant in the IR: the blowup modes corresponding to these modes are no longer tachyonic after the condensation. In fact, we find situations in which we are left after condensation with residual quotient singularities for which *all* of the chiral blowup modes correspond to *irrelevant* operators. As geometric spaces, these are *terminal* singularities with no marginal or relevant Kähler blowups,[2] well known in algebraic geometry, and correspond to non-supersymmetric string vacua with no chiral tachyons: they represent nontrivial endpoints of the renormalization group flows. Unlike the case in codimension one or two, the order in which tachyonic modes condense — the order in which we blow up — in general changes the resulting theory. Thus in general there is no canonical resolution of a given singularity. Some of the various possible distinct resolutions are related by *flip* transitions, similar to the familiar flop transitions in Calabi-Yau spaces (see figure 1).

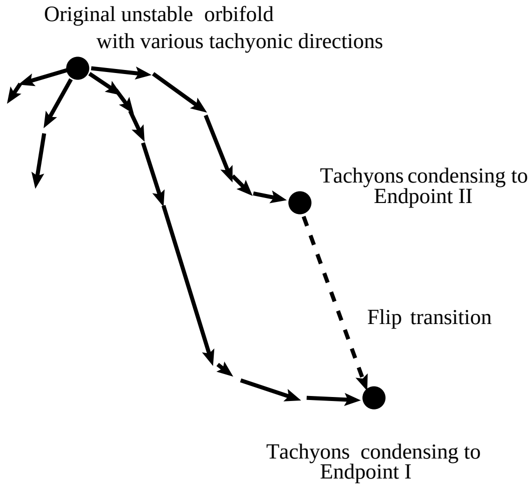

Figure 1: A heuristic picture of an unstable (UV) orbifold with several relevant directions: two distinct tachyons in $\mathbb{C}^3/\mathbb{Z}_{N(p,q)}$ condense to distinct endpoints, with a possible flip transition between them. Endpoint I stemming from condensation of a more relevant tachyon is less singular than Endpoint II (see section 4).

Thinking of renormalization flows provides a natural partial ordering of the deformation modes of a given quotient: consider a generic perturbation of the initial theory. Under RG flow, the most relevant tachyon — from the point of view of the spacetime theory, this is the state with the most negative mass squared — will condense most rapidly. For some range of initial values, we can obtain an approximate picture of the actual RG tra-

[2]Note that by Hironaka's famous theorem on resolution of singularities [8], there are always Kähler blowup modes, but most of these are irrelevant [9].

jectory (our surrogate for the time evolution) by imagining that we first follow the most relevant tachyon to a fixed point: in general under the RG flow the original singularity splits into several singular points,[3] which in the conformal limit, and given the noncompact (large-radius) analysis we perform here are decoupled.[4] Under this flow the conformal weights of other operators shift, and the most relevant operators remaining will again be the first to condense in each of the residual singularities. The natural prescription is thus to perform the blowup corresponding to the most relevant tachyon, then repeat the process. This is a partial ordering because the most relevant field need not be unique, and two flows corresponding to condensing fields of the same lowest dimension may lead to different endpoints. It is also worth pointing out that different decay modes (other than the most-relevant-tachyon sequence) of the original unstable orbifold are of course possible, giving rise in principle to distinct geometric endpoints.

Another subtlety in this case is the generic appearance of quotient singularities in codimension two. These occur along curves contained within the exceptional sets from the blowups, and the subsequent twisted sector states that arise are thus in some sense intermediate between localized and bulk modes. At specific points along the singular curve the singularity type changes; these quotient singularities are thus not decoupled even in the extreme infrared limit, coupled through the twisted modes propagating on the singular curve.

Following [5] we will use a toric description of the resolved quotient singularities. One of the remarkable results of that paper was the way in which the toric description encodes the algebraic structure of the chiral ring at the orbifold point. We find a similar correspondence in the cases studied here. Toric descriptions of chiral rings are known in the case of large-radius limits, but these are qualitatively different. The gauged linear sigma model allows us to interpolate smoothly between the two limits, and the relation between the two representations is an interesting question, left for future work.

Of course, localized tachyons are most interesting in the absence of bulk tachyons. As discussed in [11, 12, 3, 4, 5, 13] we can, in some cases, impose a consistent GSO projection on the $\mathcal{N} = (2,2)$ theory to obtain a modular invariant model from which the bulk tachyon has been removed, though spacetime supersymmetry is broken. In these models our goal is to approximate the generic decay. This, in general, breaks the worldsheet supersymmetry completely and is thus not accessible to our methods. Our approximation consists of a modification of the procedure mentioned above, of following the most relevant operator.

The orbifold theory is in fact invariant under three copies of the $\mathcal{N} = (2,2)$ supersymmetry algebra, and the most relevant operator is always chiral under some combination of these. We select this supersymmetry (this choice is closely linked to choosing the target space complex structure), and use it to follow the renormalization group trajectory after

[3] See e.g. [10], which uses the mirror Landau-Ginzburg description of [4] to show that under condensation of a single tachyon, a $\mathbb{C}^r/\mathbb{Z}_N$ orbifold decays into r separated orbifolds.

[4] Of course this decoupling and flattening out of the space away from the residual singularities occurs strictly only in the infinite RG-time limit: during the blowup process, i.e. during the condensation of a particular tachyon, the full space is indeed curved and does not admit any simple free field theory description.

adding this operator to the action. At each of the singularities that remain in the extreme infrared limit, we once more follow the most relevant operator chiral under the same supersymmetry. As a consistency check on the procedure, we show that the flow does not generate bulk tachyons in various classes of type-II theories: in other words, all the residual orbifold theories arising at the ends of our flows are GSO-projected.[5]

As above, this process ends when all of the singularities remaining have no relevant chiral operators. In fact, since the GSO projection removes some of the localized tachyons, one finds in general quotients that are not necessarily *terminal* singularities in the geometric sense, but appear *string-terminal*, all relevant chiral operators having been projected out. At this point, though, we once more have, in each of the decoupled theories, an enhanced supersymmetry: thus there exist generic metric blowup modes (chiral with respect to some supersymmetry) that potentially smooth out the singularities. Indeed, we find a clean combinatoric proof which shows the non-existence of type-II orbifold singularities completely devoid of *any* relevant or marginal blowup modes (Kähler or not) preserved by the GSO projection. Thus not only are there no string-terminal quotient singularities, there are in fact, for a type-II string, no terminal singularities at all. This shows that the endpoints of closed string tachyon condensation for type-II orbifold string theories in four or more noncompact dimensions are always smooth spaces.

We study the structure of the residual singularities after condensation of a single tachyon using toric methods — in particular, we use the "Smith normal form" of the toric data for the residual geometries to glean insight into their structure.

Organization: we describe the worldsheet conformal field theory of $\mathbb{C}^3/\mathbb{Z}_N$ orbifolds in section 2. Section 3 describes the representation via toric geometry of these orbifolds. Section 4 follows the renormalization group trajectories corresponding to chiral tachyon condensation in type-0 string theory: in particular we describe how this dovetails with the toric structure of $\mathbb{C}^3/\mathbb{Z}_N$ singularities, the analogs of "canonical minimal resolutions" and flip transitions therein, as well as the structure of the residual geometries obtained using the Smith normal form of the toric data thereof. Section 5 describes the situation for type-II theories and in particular discusses all-ring terminality. Two appendices A and B provide some technical details.

2. Free field theory at the orbifold point

The spectrum of states localized near a quotient singularity is tractable because, in the limit in which the localized states decouple from the bulk theory, we are effectively studying the space $\mathbb{C}^3/\Gamma$. As a conformal field theory, this is an orbifold of a free field theory, obtained by gauging the discrete symmetry group Γ. The high degree of symmetry of the free theory leads to many simplifications in the treatment of the quotient.

We will work in the RNS formulation, and study the local dynamics near a singularity of the form $\mathbb{R}^{1,3} \times (\mathbb{R}^6/\mathbb{Z}_N)$. We will choose a complex basis for the "internal" coordinates

[5] After this paper had been circulated, we learned that the corresponding analysis in the codimension two case which was begun in [13] (using the mirror Landau-Ginzburg description of [4]) has been completed in [14].

JHEP08(2004)047

such that Γ acts holomorphically. The generator is thus

$$g : (X_1, X_2, X_3) \to (\omega^{k_1} X_1, \omega^{k_2} X_2, \omega^{k_3} X_3)\,, \tag{2.1}$$

where X_i are complex coordinates on the internal space and $\omega = e^{2\pi i/N}$.

The free field theory before orbifolding enjoys an $\mathcal{N} = (8,8)$ worldsheet supersymmetry, which will be broken by the quotient (which acts as an R-symmetry). The quotient will preserve three copies of the $\mathcal{N} = (2,2)$ superconformal algebra, with supercurrents and $\mathrm{U}(1)_R$ currents

$$\begin{aligned} G_i^+ &= \psi_i^* \partial X_i \\ G_i^- &= \psi_i \partial X_i^* \\ J_i &= \psi_i \psi_i^* = i\,\partial H_i\,, \end{aligned} \tag{2.2}$$

and their antiholomorphic counterparts, where we have formed complex linear combinations of the worldsheet fermions as superpartners to the X_i. We have bosonized the U(1) current so that $\psi_i = e^{iH_i}$. With respect to this subalgebra, (X_i, ψ_i) form a chiral superfield, and g acts as a non-R symmetry with $\psi_i \to \omega^{k_i}\psi_i$.

Because of the product structure of the free theory, the spectrum of the quotient theory can be understood by working with one chiral superfield at a time. Thus, we consider the theory of one chiral superfield, and perform a $\mathbb{Z}_N$ quotient, with the action $X \to \omega^k X$ (of course, k can be set to one here, but we will want this peculiar notation later). Of particular importance to us will be the ground states in the twisted sectors: twisted-sector states are the ones that will be localized at the singular locus. The orbifold theory will have N twisted sectors and a "quantum" $\mathbb{Z}_N$ symmetry. In the j-th twisted sector, the field X satisfies

$$X(\sigma + 2\pi, \tau) = \omega^{jk} X(\sigma, \tau)\,. \tag{2.3}$$

The ground state in this sector can be shown (see [13] for a clear exposition) to be a chiral primary state (annihilated by $G^+_{-1/2}$ in addition to all positive modes) with $\mathrm{U}(1)_R$ charge $y = \{\frac{jk}{N}\}$ (the fractional part of $\frac{jk}{N}$) and conformal weight $h = \frac{y}{2}$, when $y < \frac{1}{2}$. The first excited state is antichiral, with charge $y-1$ and weight $h = \frac{(1-y)}{2}$. When $y > \frac{1}{2}$, the ground state is an antichiral state with charge $y-1$ and weight $h = \frac{(1-y)}{2}$, while the first excited state is chiral, with charge y and weight $h = \frac{y}{2}$. These exhaust the (anti)-chiral states in the theory, and the results are simply summarized by the statement that for each j we have a chiral state of charge $y/2$ and an antichiral state of charge $(1-y)/2$.

The mass-shell condition gives the mass in spacetime (i.e. the unorbifolded dimensions) of a state with R-charge q and conformal weight $h = \frac{|q|}{2}$ as

$$-\frac{\alpha'}{4} M^2 + h - \frac{1}{2} = 0\,. \tag{2.4}$$

Thus the most relevant tachyon, i.e. smallest R-charge, corresponds to the leading spacetime instability, i.e. with the most negative mass-squared.

JHEP08(2004)047

Chiral operators are of interest for several reasons. Since they saturate the inequality $h \geq \frac{|q|}{2}$ between conformal weight and $\mathrm{U}(1)_R$ charge, their operator products are nonsingular, and by taking the coincident limit produce the structure of a ring of (anti-) chiral operators. (Keeping track of both in this case is of course a bit redundant, since a chiral field in the j-th sector has a conjugate field in the $(N-j)$-th sector.) Constrained by conservation of $\mathrm{U}(1)_R$ as well as by the "quantum" $\mathbb{Z}_N$ symmetry of the orbifold theory, the structure of the ring is here particularly simple. The operator [15] creating the chiral state in the j-th sector, T_j, can be written as

$$T_j = \sigma_y e^{iy(H-\overline{H})}\,, \tag{2.5}$$

where σ_y is the bosonic twist operator (with conformal weight $h = \frac{1}{2}y(1-y)$), and ∂H the bosonized current as above. The ring is generated by

$$T = \sigma_{1/N} e^{(i/N)(H-\overline{H})}\,, \tag{2.6}$$

with

$$T_j = T^{Ny}\,. \tag{2.7}$$

There is, in addition to the identity, one chiral primary operator in the untwisted sector,

$$Y = \frac{1}{V}\psi\overline{\psi}\,, \tag{2.8}$$

the volume form of the internal space, normalized by its total volume. The two generators satisfy the relation [5]

$$T^N = Y\,. \tag{2.9}$$

The antichiral ring has a similarly simple structure. It is helpful in the sequel to note that chiral and antichiral fields under the algebra (2.2) are exchanged if we exchange $G^{\pm}$, or equivalently $(X,\,\psi)$ and $(X^*,\,\psi^*)$.

A chiral (or antichiral) field is the lowest component of an $\mathcal{N}=2$ chiral superfield whose top component can be added to the action without breaking supersymmetry. This means we can use the powerful constraints imposed by $(2,2)$ worldsheet supersymmetry to study the deformed theory, i.e. the RG flow and its endpoints. In the string theory, the most relevant operator in any sector is chiral (or antichiral), so the tractable sector includes the dominant decay modes of these unstable vacua. At any point along the renormalization group trajectory corresponding to the condensation of a chiral field, one can perform a topological twist [16] so that the flow corresponds to a family of topological theories (see [17] for the generalization to toric varieties). These, in turn, may be amenable to study using a twisted gauged linear sigma model (see e.g. [7]). In this case, one can follow the condensation of the tachyon all along the flow and not simply study its endpoints.

2.1 $\mathbb{C}^3/\mathbb{Z}_{N(p,q)}$: chiral rings

Returning to the case of interest (2.1) and recalling that the quotient theory is in fact invariant under the three copies of the $\mathcal{N}=2$ superconformal symmetry (2.2), we find eight rings of operators (anti-) chiral under each of these, in four conjugate pairs. In this section, we will focus on one of these, the (c_X, c_Y, c_Z) or the chiral ring. Furthermore we will focus largely on orbifolds that can be expressed in canonical form, i.e. $(k_1, k_2, k_3) \equiv (1, p, q)$. This includes all isolated orbifolds.

We denote noncompact $\mathbb{C}^3/\mathbb{Z}_N$ orbifolds with the geometric action on the $(X = z^4 + iz^5, Y = z^6 + iz^7, Z = z^8 + iz^9)$ target space coordinates

$$(X, Y, Z) \to (\omega X, \omega^p Y, \omega^q Z), \qquad |p|, |q| < N\,, \quad p, q \in \mathbb{Z}\,, \quad \omega = e^{2\pi i/N} \tag{2.10}$$

by $\mathbb{C}^3/\mathbb{Z}_{N(p,q)}$. String theory on such orbifolds retains no supersymmetry if $1 + p + q \neq 0 (\mathrm{mod} N)$ since the orbifold action does not lie within SU(3) — these orbifolds cannot be embedded as local singularities in a Calabi-Yau 3-fold. These are isolated singularities if p, q are coprime with respect to N. The twisted sector operators in the chiral ring of $\mathbb{C}^3/\mathbb{Z}_{N(p,q)}$

$$X_j = \prod_{i=1}^{3} X^{(i)}_{\{jk_i/N\}} = X^{(1)}_{j/N}\, X^{(2)}_{\{jp/N\}} X^{(3)}_{\{jq/N\}}\,, \qquad j = 1, 2, \ldots, N-1 \tag{2.11}$$

are constructed out of the twist fields (2.5) for each of the three complex planes parametrized by X, Y, Z. $\{x\} = x - [x]$ denotes the fractional part of x, with $[x]$ the integer part of x (the greatest integer $\leq x$).[6] By definition, $0 \leq \{x\} < 1$. This constitutes the (c_X, c_Y, c_Z) ring of twist operators, that are chiral with respect to each of the three complex planes. In the j-th twisted sector, the boundary conditions for the operators X_j are

$$X^i(\sigma + 2\pi, \tau) = \omega^{jk_i} X(\sigma, \tau)\,. \tag{2.12}$$

Based on our discussion above in the case of one chiral superfield, the chiral operators (2.11) are either the ground states or the first excited states in the various twist sectors.[7]

In this notation, we note that the orbifolds $\mathbb{C}^3/\mathbb{Z}_{N(p,q)}$, $\mathbb{C}^3/\mathbb{Z}_{N(-p,q)}$, $\mathbb{C}^3/\mathbb{Z}_{N(p,-q)}$ and $\mathbb{C}^3/\mathbb{Z}_{N(-p,-q)}$ are related by changes of complex structure implemented by the field redefinitions $Y \to Y^*$, $Z \to Z^*$ and $Y, Z \to Y^*, Z^*$ respectively. As we have seen, besides the (c_X, c_Y, c_Z) ring of operators (2.11), there are various other sets of "BPS protected" operators which comprise the other rings. It is noteworthy that e.g. the field redefinition $Z \to Z^*$ exchanges the (c_X, c_Y, c_Z) and (c_X, c_Y, a_Z) rings. One can check that the (c_X, c_Y, c_Z) ring of the $\mathbb{C}^3/\mathbb{Z}_{N(p,q)}$ orbifold is the (c_X, c_Y, a_Z) ring of the $\mathbb{C}^3/\mathbb{Z}_{N(p,-q)}$ orbifold, i.e. $\mathbb{C}^3/\mathbb{Z}_{N(p,N-q)}$ orbifold and similarly for the other rings.

As a geometric space, by convention the (c_X, c_Y, c_Z) ring of the singularity alone respects the asymptotic complex structure and geometry. In fact, twist operators in the other rings do not appear as lattice points representing blowup modes of the singularity in the toric geometric representation of the (c_X, c_Y, c_Z) ring: i.e. there is in general a different

[6] Note that for $m, n > 0$, we have $[\frac{-m}{n}] = -[\frac{m}{n}] - 1$ and therefore $\{\frac{-m}{n}\} = -\frac{m}{n} - [\frac{-m}{n}] = 1 - \{\frac{m}{n}\}$.

[7] For instance, in the sector where $\{\frac{jk_1}{N}\}, \{\frac{jk_2}{N}\} < \frac{1}{2}$, $\{\frac{jk_3}{N}\} > \frac{1}{2}$, the ground state is of the form $X^{(1)}_{j/N}\, X^{(2)}_{\{jp/N\}} (X^{(3)}_{1-\{jq/N\}})^*$ and belongs to the (c_X, c_Y, a_Z) ring, which is chiral w.r.t. X, Y and anti-chiral w.r.t. Z.

toric diagram for each of the rings so that these other rings for a given orbifold do not have an obvious interpretation in terms of its algebraic geometry. Physically, once a tachyonic chiral operator condenses, it breaks the full $\mathcal{N}=(2,2)$ worldsheet supersymmetry down to the subgroup it preserves, and the ring it belongs to. Thus if we so wish, we could, for noncompact singularities, $define$ the (c_X, c_Y, c_Z) ring to contain the most relevant tachyon. We will have occasion to describe the structure of the twist fields in all the various rings later when we discuss type-II theories and all-ring terminality.

Vertex operators belonging to the untwisted sector of these $\mathbb{C}^d/\mathbb{Z}_N$ orbifolds describe excitations propagating in the full ten dimensional spacetime while twisted sector states are localized to the singular subspace of the orbifold. The structure of the OPEs of general untwisted and twisted sector Virasoro primaries in a regulated (noncompact large volume) $V_{10-d} \to \infty$ limit shows that the bulk untwisted sector tachyon of type 0 decouples and thus can remain unexcited along RG flows associated with condensing only the localized twisted sector tachyons [5]. Thus it is sensible to study the condensation of localized tachyons.

For the nonchiral type-0 string theory, one performs a diagonal GSO projection which projects out spacetime fermions and retains the bulk tachyon. Thus all twisted sector tachyons are present (along with the untwisted tachyon). The twist operators X_j have R-charges and conformal dimensions

$$R_j \equiv \left(\frac{j}{N}, \left\{\frac{jp}{N}\right\}, \left\{\frac{jq}{N}\right\}\right) = \frac{j}{N} + \left\{\frac{jp}{N}\right\} + \left\{\frac{jq}{N}\right\}, \qquad h_j = \frac{1}{2}R_j\,. \tag{2.13}$$

The operators with $R_j < 1$ and $R_j = 1$ are relevant (tachyonic) and marginal respectively while those with $R_j > 1$ are irrelevant on the worldsheet. In addition to the X_j, there are of course the chiral primaries, $Y_i = \frac{1}{V}\psi_i\bar{\psi}_i$, i.e. the three volume forms of the internal space, normalized by its total volume. There exist relations amongst the operators X_j, Y_i.

The subset of the twisted states X_j that generates the chiral ring of $\mathbb{C}^3/\mathbb{Z}_{N(p,q)}$ in general contains more than one element (as in the case $\mathbb{C}^2/\mathbb{Z}_{N(p)}$ studied in [5]). Schematically then a given operator in the chiral ring can be decomposed into products of the generators via the ring relation $X_a \sim X_{g_1}^{m_1} X_{g_2}^{m_2} \ldots$, the m_i being integers. The R-charge of the generic twisted state X_a is given by $R_a = \sum_i m_i R_{g_i}$, where R_{g_i} is the R-charge of the generator X_{g_i}. A given operator in $\mathbb{C}^3/\mathbb{Z}_{N(p,q)}$ can be decomposed in various distinct ways so that the generator decomposition for a given operator is not unique (as in $\mathbb{C}^2/\mathbb{Z}_{N(p)}$). This non-uniqueness is fairly obvious from the toric representation of these orbifolds, which is 3-dimensional. As we will see, there is an intimate relationship between operators in the chiral ring of the orbifold and the relations amongst them and the geometry of lattice vectors in the toric representation thereof. A noteworthy fact is that the set of generators of the $\mathbb{C}^3/\mathbb{Z}_{N(p,q)}$ chiral ring in general includes irrelevant operators as well as tachyons and marginal operators. Indeed as we shall see, there exist classes of $\mathbb{C}^3/\mathbb{Z}_{N(p,q)}$ orbifolds where the entire chiral ring is generated purely by irrelevant operators, i.e. $R_{g_i} > 1$ for all the generators X_{g_i}. In such cases, there is no relevant or marginal deformation of the chiral ring and of the corresponding orbifold singularity via Kähler blowup modes.

We will now exhibit some examples elucidating the twisted states X_j with their R-charges and the generator decompositions thereof.

JHEP08(2004)047

Example $\mathbb{C}^3/\mathbb{Z}_{11}\,(1,2,7)$. The type-0 theory has tachyons $T_1 = (\frac{1}{11}, \frac{2}{11}, \frac{7}{11})$, $T_2 = (\frac{2}{11}, \frac{4}{11}, \frac{3}{11})$ with R-charges $R_1 = \frac{10}{11}$, $R_2 = \frac{9}{11}$ respectively, of which T_2 survives the chiral GSO projection to type II. The set of generators of the type-0 chiral ring consists of the tachyonic twist field operators $X_1 = (\frac{1}{11}, \frac{2}{11}, \frac{7}{11})$, $X_2 = (\frac{2}{11}, \frac{4}{11}, \frac{3}{11})$ and the irrelevant operators $X_5 = (\frac{5}{11}, \frac{10}{11}, \frac{2}{11})$, $X_6 = (\frac{6}{11}, \frac{1}{11}, \frac{9}{11})$, $X_7 = (\frac{7}{11}, \frac{3}{11}, \frac{5}{11})$, $X_8 = (\frac{8}{11}, \frac{5}{11}, \frac{1}{11})$. The remainder of the twist fields are $X_3 = (\frac{3}{11}, \frac{6}{11}, \frac{10}{11})$, $X_4 = (\frac{4}{11}, \frac{8}{11}, \frac{6}{11})$, $X_9 = (\frac{9}{11}, \frac{7}{11}, \frac{8}{11})$, $X_{10} = (\frac{10}{11}, \frac{9}{11}, \frac{4}{11})$. Then it is easy to see by comparing R-charges that the relations between various twist operators and these generators include

$$X_3 \sim X_1 X_2\,, \qquad X_4 \sim X_2^2\,, \qquad X_9 \sim X_7 X_2 \sim X_8 X_1\,, \qquad X_{10} \sim X_8 X_2 \tag{2.14}$$

and other similar expressions. The relation involving X_9 illustrates the non-uniqueness of the generator decomposition.

Example $\mathbb{C}^3/\mathbb{Z}_2\,(1,1,1)$. The only twisted sector $j = 1$ of the (c_X, c_Y, c_Z) ring has the irrelevant operator $X_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ with R-charge $R_j = \frac{3}{2}$. Thus the generator of the chiral ring consists of the single irrelevant operator X_1. It is straightforward to check that the twist fields X_j in all rings are irrelevant (as we will see in detail later when we discuss type-II theories): This is an isolated $all - ring$ terminal singularity.

Example $\mathbb{C}^3/\mathbb{Z}_N\,(1,p,-p)$. $(p,N$ coprime) The (c_X, c_Y, c_Z) ring twist field X_j has R-charge $R_j = \frac{j}{N} + \{\frac{jp}{N}\} + 1 - \{\frac{jp}{N}\} = 1 + \frac{j}{N} > 1$ so that these twisted sector states are irrelevant in conformal field theory. Thus there are no relevant or marginal operators in the (c_X, c_Y, c_Z) ring of the worldsheet string theory describing this class of singularities which thus cannot be resolved geometrically (see e.g. [18]). In general however, there are tachyonic or marginal twisted states arising from other rings (as we will see in detail later when we discuss type-II theories) so that the singularity in general is indeed resolved metrically via the nonchiral deformations.

3. $\mathbb{C}^3/\mathbb{Z}_{N(p,q)}$: toric geometry

In this section, we will sketch the toric geometry description of $\mathbb{C}^3/\mathbb{Z}_{N(p,q)}$, uniformizing our notation with the description in [5, 2] reviewed in appendix B. The description here is based on [19, 20, 17, 21, 22] (see e.g. [23] for a detailed exposition of toric geometry).

Let (x, y, z) and (u, v, w) be coordinates on $\mathbb{C}^3$ and $\mathbb{C}^3/\mathbb{Z}_{N(p,q)}$ respectively. A basis for the monomials invariant under the orbifold action is

$$u = x^N\,, \qquad v = x^{-p} y\,, \qquad w = x^{-q} z\,. \tag{3.1}$$

The ring of holomorphic functions on a neighbourhood of the noncompact $\mathbb{C}^3/\mathbb{Z}_{N(p,q)}$ orbifold singularity is generated by the monomials

$$u^{m_1} v^{m_2} w^{m_3} = x^{Nm_1 - pm_2 - qm_3}\, y^{m_2}\, z^{m_3} \tag{3.2}$$

for integer m_1, m_2, m_3. This ring is well-defined if the basis functions have positive exponents, i.e. $Nm_1 - pm_2 - qm_3 \geq 0, m_2 \geq 0, m_3 \geq 0$. The space of possible such vectors $\vec{m} = \sum_i m_i e_i$ is the cone in the $\mathbb{M}$ lattice, bounded by the vectors $e_1 = (1,0,0)$, $e_2 = (p, N, 0)$, $e_3 = (q, 0, N)$. Thus each point in the $\mathbb{M}$ lattice defines a monomial on the orbifold. The e_i form a basis for the $\mathbb{M}$ lattice.

Eq. (3.1) essentially specializes the general relation $x_i = \prod_j {z_j}^{a_{ji}}$, $i,j = 1,2,3$ between the coordinates $x_i \equiv (u,v,w) \in \mathbb{C}^3$ and $z_i \equiv (x,y,z) \in \mathbb{C}^3/\mathbb{Z}_{N(p,q)}$ (see eqs. (2.1) and (2.2) of [20]), from which we can therefore read off the a_{ij} as

$$a_{1j} \equiv \alpha_1 = (N,-p,-q)\,, \qquad a_{2j} \equiv \alpha_2 = (0,1,0)\,, \qquad a_{3j} \equiv \alpha_3 = (0,0,1)\,, \qquad j = 1,2,3\,, \tag{3.3}$$

i.e. the matrix a_{ij} is formed by juxtaposing the rows α_1, α_2, α_3. Since the orbifold acts as $g : z_j \mapsto \omega^{g_j} z_j$ with rational g_j, we have $\sum_j g_j a_{ji} \in \mathbb{Z}$, $i = 1,2,3$.

The vectors α_i are constructed orthogonal to the basis $e_i \in \mathbb{M}$: specifically $\alpha_1 \perp e_2, e_3$, $\alpha_2 \perp e_1, e_3$, $\alpha_3 \perp e_1, e_2$. They form an integral basis for the $\mathbb{N}$ lattice dual to $\mathbb{M}$. (3.3) gives the vertices of the simplex Δ defining the fan of cones subtended with the origin $0 = (0,0,0)$ as the apex in $\mathbb{N}$. Alternatively one may of course choose to begin with a cone in the $\mathbb{N}$ lattice and construct from it the dual lattice $\mathbb{M}$ as the space of monomials.

Subcones of the cone $C(0; \alpha_1, \alpha_2, \alpha_3)$ which have (real) dimension p determine "toric" codimension-p algebraic subspaces of $\mathbb{C}^3/\mathbb{Z}_{N(p,q)}$ with (complex) codimension p. Thus toric divisors (which are algebraically embedded codimension one subspaces) are determined by 1-dimensional cones, i.e. rays in $C(0; \alpha_1, \alpha_2, \alpha_3)$.

Note that the cone $C(0; \alpha_1, \alpha_2, \alpha_3)$ has volume[8] N in terms of the units defined by the lattice $\mathbb{N}$. The relationship between the group action and the lattice $\mathbb{N}$ in our case is visible in the basis in eq. (3.3). In general, a cone with volume $k > 1$ corresponds to an orbifold singularity by a discrete group Γ whose order is k. (If cyclic, this is automatically a $\mathbb{Z}_k$ singularity.) To determine the group Γ, one needs a basis for the lattice $\mathbb{N}$ which is nicely adapted to the group action, in a manner analagous to eq. (3.3). In fact, $\mathbb{N}$ is a lattice of rank 3 contained in the standard lattice $\mathbb{Z}^3$, and the group Γ is the quotient $\mathbb{Z}^3/\mathbb{N}$, which is a finite group. The fact that a finitely generated abelian group is a direct sum of cyclic groups is a standard theorem in algebra (see, for example, [24]), and a specific algorithm to calculate the sum of cyclic groups — the "Smith normal form" — will be presented in section 4.

The simplex Δ is the intersection of the fan with an affine hyperplane passing through the three vertices α_i. The equation describing the affine hyperplane containing the simplex Δ takes the form $\ell_\Delta(x,y,z) = 1$, where

$$\ell_\Delta(x,y,z) = \left(\frac{1+p+q}{N}\right) x + y + z\,. \tag{3.4}$$

We will refer to this affine hyperplane also as Δ. The normal to the hyperplane Δ is the vector $v_\perp = (1+p+q, N, N)$, which satisfies $\alpha_i \cdot v_\perp = N > 0$. The left side of figure 2 shows the affine hyperplane and the simplex Δ within it.

There is a remarkable correspondence between operators in the orbifold conformal field theory and subspaces in the $\mathbb{N}$ lattice. A given orbifold conformal field theory has relevant/marginal/irrelevant operators that correspond to specific lattice points in $\mathbb{N}$. In our normalization, the linear function ℓ_Δ evaluated on a specific lattice point j yields a value $\Delta_j = \ell_\Delta(x_j, y_j, z_j)$ which is *equal* to the R-charge R_j of the corresponding operator.

[8] The volume of the cube generated by three 3-vectors is $V(\alpha_1, \alpha_2, \alpha_3) = |\det(\alpha_1, \alpha_2, \alpha_3)| = |\alpha_1 \cdot \alpha_2 \times \alpha_3|$.

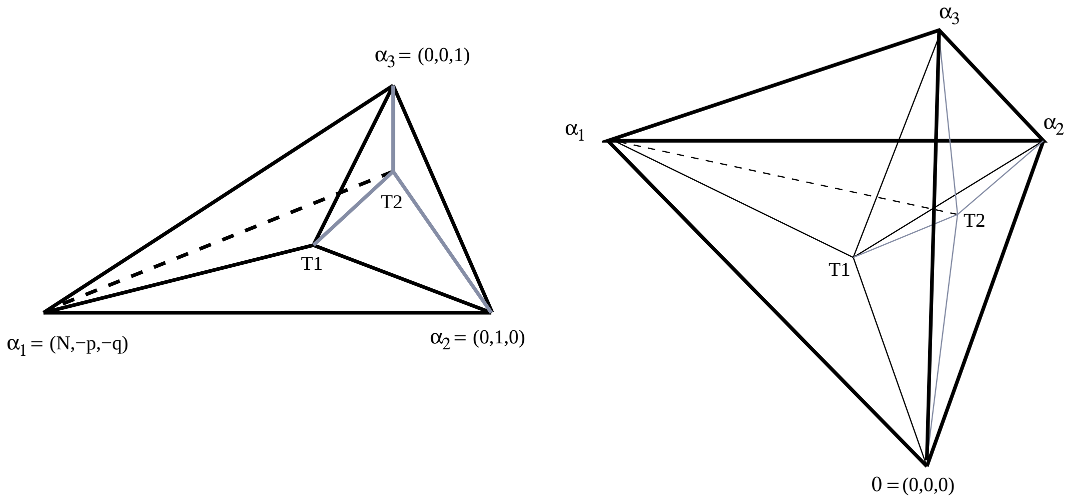

Figure 2: The fan of cones for the $\mathbb{N}$ lattice of $\mathbb{C}^3/\mathbb{Z}_{N(p,q)}$, with the vertices of the simplex Δ, as well as tachyons T_1, T_2 in the interior of the cone and the corresponding subdivisions thereof. (The figure on the left shows the simplex and its subdivision; the figure on the right shows the actual cones in the fan.)

A given lattice point $P_j = (x_j, y_j, z_j)$ can then be translated to an twisted sector operator as follows: realize that this vector can be expressed in the $\{\alpha_1, \alpha_2, \alpha_3\}$ basis as

$$(x_j, y_j, z_j) = a\alpha_1 + b\alpha_2 + c\alpha_3\,. \tag{3.5}$$

This then corresponds to an operator O_j with R-charge

$$R_j \equiv (a, b, c) = \left(\frac{x_j}{N},\ y_j + x_j\frac{p}{N},\ z_j + x_j\frac{q}{N}\right). \tag{3.6}$$

Conversely, an operator O_j with R-charge $R_j = (\frac{j}{N}, \{\frac{jp}{N}\}, \{\frac{jq}{N}\})$ corresponds to a lattice point $P_j = (j, -[\frac{jp}{N}], -[\frac{jq}{N}])$. In general, there are lattice points lying "above" the affine hyperplane Δ that correspond to irrelevant deformations. These have $R_j = \Delta_j > 1$.

Conformal field theories corresponding to supersymmetric $\mathbb{C}^3/\mathbb{Z}_N$ orbifolds always have marginal deformations that are represented by points β_k that lie *on* the affine hyperplane Δ. Thus $R_j = \Delta_j = 1$ for the β_k and we will refer to the affine hyperplane Δ as the plane of marginal operators. The toric picture of blowing up a supersymmetric orbifold by a marginal operator then consists of adding to the cone an irreducible divisor (i.e. a ray) corresponding to the marginal operator β_k and triangulating Δ into smaller sub-simplices that each have β_k as a vertex. Such a subdivision of Δ by one or more of the β_k gives a subdivision of the cone into subcones, corresponding to a blowup that gives a (partial) resolution of the orbifold singularity. Subdividing maximally using all the marginal operators (i.e. all the blowup modes) present resolves the orbifold completely so that the resulting space is smooth. See e.g. figure 5 of [20] for a picture of the supersymmetric $\mathbb{C}^3/\mathbb{Z}_{11}$ $(1, 3, -4)$ orbifold. In general, there are multiple distinct ways to subdivide using the β_k in codimension three, which give resolved spaces of different topologies: these are related by flop transitions. Since the β_k lie on Δ, the maximal subdivision always yields N minimal subcones each of volume one, making up the volume N of the original cone.

In the nonsupersymmetric cases we study here, in addition to possible β_k that lie *on* Δ corresponding to marginal deformations, there are points T_k in the interior of the cone (i.e. "below" the affine hyperplane Δ) that correspond to relevant (tachyonic) deformations. These have $R_j = \Delta_j < 1$.

As we have seen in the previous section, the chiral ring of twist field operators is generated by a subset comprising relevant, marginal and irrelevant operators. From the point of view of the toric representation we have described here, the relations $X_a \sim X_{g_1}^{m_1} X_{g_2}^{m_2} \dots$ between various twist operators and the generators (see e.g. (2.14)) are precisely equivalent to the different possible ways to generate a given lattice vector by adding lattice vectors corresponding to generators.

The cone corresponding to a nonsupersymmetric singularity can be subdivided by the points T_k as well as the β_k. Figure 2 shows tachyons T_1 and T_2 and their corresponding subdivisions. Such a tachyonic subdivision by a T_k results in a partially-resolved space with total subcone-volume less than N, the volume of the original cone: in other words, the resulting space is less singular than the original orbifold. In general, if there are multiple tachyonic points T_k in the interior, there are multiple distinct ways to subdivide which correspond to distinct resolutions of the original singularity, typically with distinct total volumes of the subcones. In other words, there is in general no canonical resolution. Some of these distinct resolutions are related by what are known as *flip transitions.* On the other hand, distinct subdivisions which give identical total volumes of their corresponding subcones are potentially related by flop transitions. In general the subcones obtained after all the subdivisions have been executed need not have volume $V_{\text{subcone}} = 1$ each: the endpoint of maximal subdivisions corresponding to a generic collection of tachyons in $\mathbb{C}^3/\mathbb{Z}_{N(p,q)}$ includes subcones of volume $V_{\text{subcone}} > 1$. Such a subcone corresponds to a *geometric* terminal singularity, with no further lattice points on its affine hyperplane Δ_{subcone} or in the interior thereof: thus there are no further relevant or marginal chiral deformations of the corresponding conformal field theory by which it can be resolved via Kähler blowups. In the next section, we will elaborate more on these phenomena.

4. Geometric terminal singularities, flip transitions and all that

Unlike complex codimension one and two orbifold singularities, nonsupersymmetric (codimension three) $\mathbb{C}^3/\mathbb{Z}_{N(p,q)}$ orbifolds generically include "geometric" terminal singularities, containing no marginal or relevant Kähler blowup modes — a phenomenon well-known in algebraic geometry [25]. In physical terms, the corresponding worldsheet string conformal field theories do not contain any chiral relevant or marginal twisted sector operator by which the singularities can be resolved. A simple example is $\mathbb{C}^3/\mathbb{Z}_2$ $(1,1,1)$: the twisted state from the only sector $j=1$ has R-charge $R_j = \frac{3}{2} > 1$, corresponding to an irrelevant operator in conformal field theory. More generally, it is easy to see that $\mathbb{C}^3/\mathbb{Z}_{N(p,-p)}$ is a geometric terminal singularity for p, N coprime: the R-charge of the j-th twisted state is $R_j = 1 + \frac{j}{N} > 1$, so that the (c_X, c_Y, c_Z) (chiral) twist fields in each twisted sector correspond to irrelevant operators in the conformal field theory.[9]

[9] In the supersymmetric $\mathbb{C}^3/\mathbb{Z}_N$ cases, there always exist marginal chiral deformations which resolve

This result is further strengthened by the fact that toric $\mathbb{C}^3/\mathbb{Z}_N$ orbifold singularities have no complex structure deformations [26], unlike $\mathbb{C}^2/\mathbb{Z}_N$: resolutions must be by Kähler blowup modes.

On the other hand, sometimes the singularities can indeed be resolved. A simple example that is devoid of the intricacies to follow later is:

Example $\mathbb{C}^3/\mathbb{Z}_N(1,1,1)$. Figure 3 shows the vertices $(N,-1,-1),(0,1,0),(0,0,1)$ of the cone in the toric diagram. For $N=3$, this can be recast as the familiar supersymmetric $\mathbb{Z}_3$ orbifold, while for $N>3$ there is a single tachyonic operator $T_1=(1,0,0)$ with R-charge $R_j=\frac{3}{N}$ that generates the chiral ring (shown in the figure). The operators $T_j=(j,0,0)$ with R-charges $R_j=\frac{j}{N}$ are tachyonic for $j<N$ and are generated by T_1: the T_j are collinear with the generator T_1 and the origin. As can be seen from figure 3, the three subcones arising from the subdivision by T_1 have volumes $V(0;\alpha_1,\alpha_2,T_1)=V(0;\alpha_2,\alpha_3,T_1)=V(0;\alpha_3,\alpha_1,T_1)=1$ for any N, so that condensation of the most relevant tachyon T_1 here leads to a resolution of the orbifold to flat space.

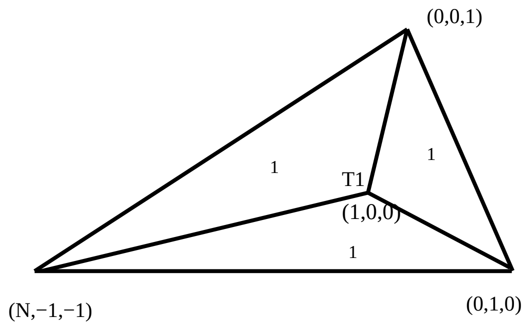

Figure 3: $\mathbb{C}^3/\mathbb{Z}_N$ $(1,1,1)$: The three points defining the affine hyperplane Δ are shown, along with the sole tachyon T_1 and its corresponding subdivision.

4.1 Minimal resolutions, terminal singularities, and flips

The traditional mathematical approach to studying resolutions of singularities (orbifold or otherwise) proceeds in two steps. First, thanks to a famous theorem of Hironaka [8], it is known that there is a sequence of (Kähler) blowups which yields a smooth space.[10] However, this resolution is far from unique, because it is possible to perform arbitrary additional blowups along smooth subvarieties of the smooth space, yielding other resolutions. The question thus arises: can we find some kind of "minimal" resolution of a singularity?

In complex dimension one, further blowups do not change the space and the question does not arise. In complex dimension two, the existence of a unique minimal resolution of singularities follows from the results of the classical Italian school of algebraic geometry (from the first half of the twentieth century). The analogous question in complex dimension three was the subject of intense efforts by mathematicians in the 1980's.

When studying nonlinear sigma models, the possibility of arbitrary blowups along smooth subvarieties is not an issue for the physical theory, because such blowups correspond

the singularity to flat space. On the other hand, in supersymmetric $\mathbb{C}^4/\mathbb{Z}_N$ (no tachyons) and higher dimensions, there exist singularities which cannot be resolved by marginal chiral deformations either: the proof of this uses toric techniques combined with results from refs. [26, 27].

[10]Traditionally in algebraic geometry, the question of Kähler metrics on these blowups was not considered, but since the blowups are projective morphisms, Kähler metrics on them can be chosen so that they indeed become Kähler blowups.

to irrelevant operators in the sigma model [9]. In fact, the classification of sigma model operators into relevant, marginal, and irrelevant exactly parallels the criterion introduced by Mori [28] to attack the minimal resolution question: Mori studied the sign of the intersection numbers $K_X \cdot C$, where K_X is the algebraic divisor class measuring the zeros and poles of a holomorphic n-form, and C is an arbitrary algebraic curve on X. When $K_X \cdot C < 0$, the sigma model operator which creates C via a blowup is irrelevant; when $K_X \cdot C = 0$ the operator is marginal; and when $K_X \cdot C > 0$ the operator is relevant.

Mori showed that starting from a smooth complex threefold, if there are any C's with $K_X \cdot C < 0$, then there is always a blowdown map which shrinks some of those C's to zero size. (This is precisely what we would expect from analyzing the sigma model, since such C's correspond to irrelevant operators.) In complex dimension two, the blowdown yields a smooth surface, but in complex dimension three, the blowdown can introduce singularities, the so-called *terminal singularities*. By definition (see, e.g., [25]), these are singularities for which some tensor power $(\Omega^n)^{\otimes r}$ of the sheaf of holomorphic n-forms has local sections, with the property that for every blowup, the pullback of any local section of $(\Omega^n)^{\otimes r}$ vanishes along each divisor created by the blowup. In the case of orbifold singularities, this rather technical condition can be replaced by a simpler condition (which is equivalent to checking for relevant or marginal tachyons), and the terminal orbifold singularities in complex dimension three can be completely classified [29].

Later study revealed that a minimal resolution could not be reached by simply following a sequence of blowdowns from the starting resolution: in addition, one needs to consider "flips," in which (locally) a single curve C on X shrinks to zero size, but then a different blowup is done which causes a new curve C^+ to grow, creating the space X^+. We have $K_X \cdot C < 0$ but $K_{X^+} \cdot C^+ > 0$. (Both spaces X and X^+ have terminal singularities — there are no flips which just involve smooth spaces.) Work by Reid, Kawamata and others culminated in the final result by Mori [30] which showed that minimal resolutions exist and can be obtained from arbitrary resolutions by a sequence of blowdowns and flips.

In fact, the theorem had earlier been proven in the case of toric blowups, blowdowns, and flips by Danilov [31]. The combinatorics of this process on toric varieties is the tool we are using here.

4.2 Relevance of tachyons, volume minimization and flips

In general, for the orbifolds being studied in this paper, there are multiple tachyons not collinear with the origin in the interior of the cone with distinct R-charges i.e. order of relevance, thus giving rise to distinct subdivisions and potential flip (or flop) transitions between the distinct subdivisions. As we have mentioned before, the most relevant tachyon (smallest R-charge) is the dominant perturbation of the worldsheet theory, and correspondingly the dominant unstable direction from the point of view of the target spacetime (most negative M^2). Thus the most relevant tachyon is most likely to condense, followed by the next most relevant tachyon and so on. In the $\mathbb{C}^2/\mathbb{Z}_{N(p)}$ cases, the Hirzebruch-Jung minimal resolution theory ensures that the final endpoint of this sequence of most relevant tachyons is a set of decoupled flat space regions (see appendix B). $\mathbb{C}^3/\mathbb{Z}_{N(p,q)}$ however has richer structure: the generic sequence of tachyonic perturbations leads to geometric terminal singularities as we have mentioned above. Furthermore, tachyons with different degrees of "tachyonity", i.e. relevance of R-charge, give rise with different degrees of likelihood to resolutions which generically have distinct topologies. There is an interesting calculation that shows the relation between the relevance (R-charges) of distinct tachyons and the volumes of the subcones resulting from the subdivisions corresponding to those tachyons. Let us begin with the cone $C(0;\alpha_1,\alpha_2,\alpha_3)$ corresponding to the unresolved $\mathbb{C}^3/\mathbb{Z}_{N(p,q)}$ singularity. Now add a lattice point $T_j = (j, -[\frac{jp}{N}], -[\frac{jq}{N}])$ corresponding to a tachyon T_j. Then the total volume of the three residual subcones after subdivision with the tachyon T_j is easily calculated: let $V(e_0;e_1,e_2,e_3)$ denote the volume of a cone subtended by the vectors e_1,e_2,e_3 with the vector e_0 being the apex. Then (see e.g. figure 2) we have

$$\begin{aligned} V_{\text{subcones}}(T_j) &= V(0;\alpha_1,\alpha_2,\alpha_3) - V(T_j;\alpha_1,\alpha_2,\alpha_3) \\ &= \alpha_1\cdot(\alpha_2\times\alpha_3) - \Big[(\alpha_1 - T_j)\cdot(\alpha_2 - T_j)\times(\alpha_3 - T_j)\Big] = N - N(1-R_j) \\ &= NR_j < N\,. \end{aligned} \tag{4.1}$$

Now if we consider two tachyons T_1 and T_2 (as in figure 2), such that T_1 is more relevant than T_2, i.e.

$$R_1 < R_2\,, \qquad R_i = \frac{j_i}{N} + \left\{\frac{j_i p}{N}\right\} + \left\{\frac{j_i q}{N}\right\}. \tag{4.2}$$

This implies

$$V_{\text{subcones}}(T_1) < V_{\text{subcones}}(T_2)\,. \tag{4.3}$$

This shows that given a singularity, condensation of a more relevant tachyon locally leads to a less singular resolution. For instance, imagine that the less relevant tachyon T_2 begins to condense (figure 1): if left unperturbed, the singularity will decay and settle down to the resolution with three decoupled sub-singularities with total volume $V_{\text{subcones}}(T_2)$ of the three corresponding subcones. However, in the process of decay, if a small fluctuation causes condensation of the more relevant T_1, this will dominate over the earlier blowup causing a *flip* transition that dynamically forces the singularity to settle down to the less singular resolution with volume $V_{\text{subcones}}(T_1) < V_{\text{subcones}}(T_2)$. From the point of view of the toric diagram, condensing the tachyon T_j by blowing up the corresponding divisor is associated with adding the corresponding lattice point (and the ray thereof) and subdividing thereby. Likewise blowing down a divisor corresponds to removing its associated ray and the subdivision thereof. In this language, a flip transition can be thought of in a manner similar to a flop: as a blowdown combined with a blowup, except that in this case there is an associated potential hill owing to the fact that relevant operators are at work here (we remind the reader that a flop consists of truly marginal deformations lying on the moduli space of supersymmetric string vacua). For instance in figure 1, endpoint 1 is less singular than endpoint 2: thus there is a potential hill for the flip transition $2\to 1$.

For either resolution, the residual subcones generically have further tachyons themselves, i.e. the resulting system is itself unstable to tachyon condensation. Thus iterating the above argument sequentially suggests that sequential condensation of the most relevant tachyons (instead of condensation of generic tachyons) is what leads to a decoupled set of residual singularities that is as flat as possible, i.e. with minimal total volume $V_{\rm subcones}$ for the entire set of subcones. However the local argument for the inequality above does not shed light on the global question of whether sequentially condensing the most relevant tachyon indeed leads to the least singular endpoint, i.e. whether the resulting stable singularity (with no further tachyons) has minimum total volume $V_{\rm subcones}$.

4.3 The Smith normal form

Let us now ask what the structure of the residual singularities is. For instance (see figure 2) we obtain the three decoupled subcones $C(0; T_1, \alpha_1, \alpha_2), C(0; T_1, \alpha_2, \alpha_3), C(0; T_1, \alpha_3, \alpha_1)$ as the endpoint of the condensation of the first tachyon, say, T_1. Each of these residual subcones represents a new orbifold conformal field theory (decoupled from the other theories corresponding to the other subcones) which itself generically contains further tachyons. In general, such a residual subcone, defined by three lattice points, is an orbifold singularity whose associated group is determined by the lattice points. Abstractly, it is easy to say what the group is: the three lattice points generate a subgroup $\mathbb{M}$ of the standard lattice $\mathbb{Z}^3$, and the quotient group $\mathbb{Z}^3/\mathbb{M}$ coincides with the orbifold group Γ. By a standard theorem in abstract algebra [24], such a quotient group can always be written as a direct sum of cyclic groups.

Concretely, the group Γ associated with such a singularity can be calculated efficiently by an algorithm which puts an integer matrix into the so-called Smith normal form. The integer matrix in question describes the inclusion of $\mathbb{M}$ into $\mathbb{Z}^3$ (via a choice of generators), and the Smith normal form algorithm produces a new choice of generators which makes the structure as a direct sum of cyclic groups manifest (as well as indicating the group action for each of the summands). This is best illustrated by an example. For a given subcone, say $C(0; T_1, \alpha_2, \alpha_3)$, consider the matrix formed by juxtaposing the column vectors T_1, α_2, α_3 representing the subcone vertices and performing the following set of elementary row and column operations on it to eventually obtain a diagonal matrix

$$\begin{pmatrix} j_1 & 0 & 0 \\ -[\frac{j_1 p}{N}] & 1 & 0 \\ -[\frac{j_1 q}{N}] & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} j_1 & 0 & 0 \\ -[\frac{j_1 p}{N}] & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} j_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.4}$$

We have performed the operations $T_1 + [\frac{j_1 p}{N}]\alpha_2$ and $T_1 + [\frac{j_1 p}{N}]\alpha_2 + [\frac{j_1 q}{N}]\alpha_3$ on the first column, in the first and second step here. This then corresponds to the singularity $\mathbb{Z}_{j_1}$, with action $(1, [\frac{j_1 p}{N}], [\frac{j_1 q}{N}])$ on the three coordinates represented by the vertices T_1, α_2, α_3.

More generally, begin with the matrix formed by juxtaposing the column vectors $v_1, v_2, v_3 \in \mathbb{Z}^3$ representing the vertices of a given subcone $C(0; v_1, v_2, v_3)$ and perform elementary row and column operations[11] to obtain a diagonal matrix diag$[d_1, d_2, d_3]$, with $d_3|d_2|d_1$, as the Smith normal form of the original matrix: this corresponds to a $\mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \times \mathbb{Z}_{d_3}$ singularity.[12] The Smith normal form algorithm[13] essentially executes $\mathrm{GL}(3, \mathbb{Z})$

[11]These consist of: (1) multiplying a row (column) by (-1), (2) adding an integral multiple of one row (column) to another, (3) interchanging rows (columns).

[12]The fact that this can always be done is the well-known structure theorem for finitely presented abelian groups [24].

[13]It is useful to note that the command **ismith** in Maple performs the Smith algorithm.

transformations[14] on the lattice vectors as well as on the lattice basis itself: column operations change the lattice vectors while row operations are simply a change of basis of the lattice which thus do not affect the relations between the lattice vectors (and therefore the structure of the orbifold action). Note that generically both column and row operations are required to obtain the Smith normal form, putting the lattice vectors describing the orbifold into canonical form. As another example, consider the following matrix which appears for the subcone $C(0; T_2, \alpha_1, \alpha_2)$ in $\mathbb{C}^3/\mathbb{Z}_{11}$ $(1, 2, 7)$

$$\begin{pmatrix} 11 & 0 & 2 \\ -2 & 1 & 0 \\ -7 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 11 & 0 & 2 \\ 0 & 1 & 0 \\ -7 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} -3 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (4.5)$$

We have performed the column operations $\alpha_1 + 2\alpha_2$ and $\alpha_1 + 2\alpha_2 - 7T_2$ on the first column, in the first and second step here, followed by a row operation $r_1 + 2r_3$ in the third step. This thus corresponds to a $\mathbb{Z}_3\,(-7, 1, 2)$ singularity (the orbifold actions are on T_2, α_1, α_2 respectively), which is the same as $\mathbb{Z}_3\,(-1, 1, 2)$. We will find more intricate use for this algorithm in what follows.

There is an alternate equivalent way of realizing the orbifold action, which is to find a linear combination of the vertices in question that is itself a vector in the lattice generated by the vertices. As an example, it is clear from our first example above that the vector $\frac{1}{j_1}(T_1 + [\frac{j_1 p}{N}]\alpha_2 + [\frac{j_1 q}{N}]\alpha_3) = (1, 0, 0)$ is clearly a lattice point itself: from this we read off the orbifold action to be $\mathbb{Z}_{j_1}$ $(1, [\frac{j_1 p}{N}], [\frac{j_1 q}{N}])$ as above. In the second example above, we have $-\frac{1}{3}(-7T_2 + \alpha_1 + 2\alpha_2) = (1, 0, 0)$ as a lattice vector, giving $\mathbb{Z}_3\,(-7, 1, 2) \equiv \mathbb{Z}_3\,(-1, 1, 2)$ as above.

It is important to note that the vector that arises as the linear combination in the Smith normal form is only defined up to linear combinations of integral multiples of the original lattice vectors.

4.4 Renormalization of subsequent tachyons within subcones

As mentioned previously, a tachyonic chiral operator that condenses breaks the full $\mathcal{N} = (2, 2)$ worldsheet supersymmetry down to the subgroup preserved by the ring it belongs to (say (c_X, c_Y, c_Z)). Furthermore, the residual theories arising at the endpoint of condensation of a given tachyon have different R-symmetries from the original conformal field theory. Thus the R-charges of the subsequent tachyons remaining in the residual geometries get "renormalized" after a given tachyon has condensed, the specific renormalization of a particular subsequent tachyon depending on which of the three decoupled subcones it lies within. A subsequent tachyon in a residual subcone is a chiral operator with respect to the R-symmetries preserved by that subcone. In figure 2 for instance, the subsequent tachyon T_2 lies in the subcone $C(0; \alpha_2, \alpha_3, T_1)$. It is then easy to see that the R-charge of T_2 is in general different, since its geometry relative to the affine hyperplane Δ_{subcone} (the plane passing through the subcone vertices T_1, α_2, α_3) is different from before. Since Δ_{subcone}

[14] $\mathrm{GL}(3, \mathbb{Z})$ consists of 3×3 matrices with $Det = \pm 1$: the determinant condition is equivalent to requiring the entries of the inverse matrix to be integral.

dips inwards, in other words away from the original $\Delta_{\rm cone}$, a subsequent tachyon is always closer to $\Delta_{\rm subcone}$ (corresponding to the subcone it lies within). Thus the renormalized R-charges of subsequent tachyons are always greater than their prior values.

This renormalization of the R-charge can be calculated by studying the geometry of the subsequent tachyon in question relative to the subcone it lies within (the expressions appearing below can be checked for agreement with $R'_k = a+b+c$ calculated for T_k lying in a subcone in the next section on type-II theories). The equation describing an affine hyperplane $\Delta_{\rm subcone}$ passing through three lattice points $v_1 = (x_1, y_1, z_1), v_2 = (x_2, y_2, z_2), v_3 = (x_3, y_3, z_3)$ can be written as

$$\ell_{\Delta_{\rm subcone}}(x,y,z) \equiv \frac{D(x,y,z)}{D(x_1,y_1,z_1)} = 1\,, \qquad D(x,y,z) = \det \begin{pmatrix} x & y & z \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{pmatrix}. \tag{4.6}$$

This has been normalized so that a lattice point lying on $\Delta_{\rm subcone}$ corresponds to a marginal operator with R-charge $R_j = \ell_{\Delta_{\rm subcone}}(x_j, y_j, z_j) = \frac{D(x_j,y_j,z_j)}{D(x_1,y_1,z_1)} = 1$. To illustrate this expression, consider the subsequent tachyon T_2 in the residual subcone $C(0; T_1, \alpha_2, \alpha_3)$ in figure 2. Then the renormalized R-charge of T_2 after T_1 has condensed can be read off from the equation describing $\Delta_{\rm subcone}$ through T_1, α_2, α_3 (simplifying the above expression using (2.13) for the R-charge) as

$$R'_2 = \ell_{\Delta^{23T}}(T_2) \equiv \frac{D^{23T}(T_2)}{D^{23T}(T_1)} = R_2 + \frac{j_2}{j_1}\Big(1-R_1\Big) = R_2\left[1 + \frac{j_2}{j_1}\,\frac{(1-R_1)}{R_2}\right]. \tag{4.7}$$

Similarly, the corresponding expressions for the renormalized R-charge if the subsequent tachyon T_2 lies within the residual subcones $C(0; T_1, \alpha_1, \alpha_2)$ or $C(0; T_1, \alpha_3, \alpha_1)$ in figure 2 are

$$\begin{aligned} R'_2 = \ell_{\Delta^{12T}}(T_2) &\equiv \frac{D^{12T}(T_2)}{D^{12T}(T_1)} = R_2\left[1 + \frac{\{\frac{j_2 q}{N}\}}{\{\frac{j_1 q}{N}\}}\,\frac{(1-R_1)}{R_2}\right], \qquad T_2 \in (0; T_1, \alpha_1, \alpha_2)\,, \\ R'_2 = \ell_{\Delta^{31T}}(T_2) &\equiv \frac{D^{31T}(T_2)}{D^{31T}(T_1)} = R_2\left[1 + \frac{\{\frac{j_2 p}{N}\}}{\{\frac{j_1 p}{N}\}}\,\frac{(1-R_1)}{R_2}\right], \qquad T_2 \in (0; T_1, \alpha_3, \alpha_1)\,. \end{aligned} \tag{4.8}$$

It is easy to see that since we are considering tachyons, we have $R_1 < 1$ so that $R'_2 > R_2$ always. In the examples below, we will illustrate this fact with several subsequent tachyons that become marginal or near-marginal or irrelevant.

We now present a type-0 example describing the tachyons therein and the toric blowups thereof. It is useful to keep in mind the supersymmetric $\mathbb{C}^3/\mathbb{Z}_{11}\,(1,3,-4)$ orbifold that is studied in some detail in [20].

Example $\mathbb{C}^3/\mathbb{Z}_{13}\,(1,2,5)$**.** See figure 4. The most relevant tachyon in the type-0 theory in this example lies in the (c_X, c_Y, c_Z) ring and is $T_1 = (\frac{1}{13}, \frac{2}{13}, \frac{5}{13})$ with R-charge $R_1 = \frac{8}{13}$. This ring also has the tachyons $T_3 = (\frac{3}{13}, \frac{6}{13}, \frac{2}{13})$ and $T_8 = (\frac{8}{13}, \frac{3}{13}, \frac{1}{13})$ with R-charges $R_3 = \frac{11}{13}$ and $R_8 = \frac{12}{13}$ respectively. In the toric fan (with vertices $\alpha_1 = (13, -2, -5)$, $\alpha_2 = (0, 1, 0)$, $\alpha_3 = (0, 0, 1)$), these tachyons correspond to the lattice vectors $T_1 = (1, 0, 0)$, $T_3 = (3, 0, -1)$, $T_8 = (8, -1, -3)$. As can be seen from the volumes below (or otherwise),

T_1 and T_3 are coplanar with α_3 while T_3 and T_8 are coplanar with α_1. For convenience, we note down here the volumes of some subcones that arise in the subdivisions we will describe below

$$\begin{aligned}
&V(\alpha_1, T_3, T_8) = V(\alpha_3, T_1, T_3) = 0\,, \qquad V(\alpha_2, \alpha_3, T_1) = V(T_3, T_1, T_8) = V(\alpha_1, T_1, T_8) = 1\,,\\
&V(\alpha_1, \alpha_2, T_3) = V(\alpha_1, \alpha_3, T_1) = 2\,, \qquad V(\alpha_1, \alpha_3, T_8) = 3\,, \qquad V(\alpha_1, \alpha_2, T_1) = 5\,,\\
&V(\alpha_3, T_1, T_8) = V(\alpha_1, \alpha_2, T_8) = V(\alpha_2, T_3, T_8) = V(\alpha_2, T_1, T_3) = 1
\end{aligned} \tag{4.9}$$

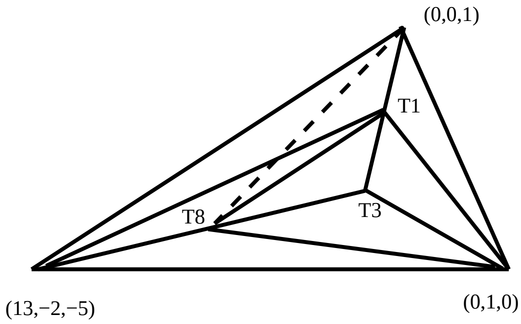

Figure 4: $\mathbb{C}^3/\mathbb{Z}_{13}(1, 2, 5)$: The three points defining the affine hyperplane Δ are shown, along with the three tachyons T_1, T_3, T_8 and two distinct sequences of subdivisions. The solid lines correspond to the sequence of most relevant tachyons.

Let us first consider the subdivisions shown by the solid lines which correspond to sequentially blowing up the most relevant tachyon, i.e. the sequence T_1, T_3, T_8: this gives the total volume of the subcones to be $6(1) + 2$, which is smaller and thus will dynamically give rise to a flip transition. Let us consider this sequence in more detail: the first subdivision by T_1 gives the three subcones $C(0; \alpha_3, T_1, \alpha_2)$, $C(0; \alpha_1, T_1, \alpha_3)$, $C(0; \alpha_1, \alpha_2, T_1)$, which by the Smith normal form are respectively flat space, $\mathbb{Z}_2$ and $\mathbb{Z}_5(1, 2, -3)$ singularities. Alternatively it is straightforward to realize the combinations $\frac{1}{5}(-\alpha_1 - 2\alpha_2 + 13T_1) = (0, 1, 0)$ and $\frac{1}{2}(-\alpha_1 - 5\alpha_3 + 13T_1) = (0, 1, 0)$ as vectors in the lattice: these imply that the subcones $C(0; \alpha_1, \alpha_2, T_1)$ and $C(0; \alpha_1, T_1, \alpha_3)$ are respectively $\mathbb{Z}_5(1, 2, -13) \equiv \mathbb{Z}_5(1, 2, -3)$ and $\mathbb{Z}_2(1, 5, -13) \equiv \mathbb{Z}_2(1, 1, -1)$ singularities (shifting by multiples of the corresponding lattice vectors). Thus the three decoupled residual geometries after this subdivision by the most relevant tachyon T_1 include a terminal singularity along with a supersymmetric orbifold with only marginal blowup modes (i.e. no relevant blowup modes). Using (4.7) and (4.8), we can calculate the renormalized R-charges of the subsequent twisted states as

$$R'_3 = \ell_{\Delta^{12T}}(T_3) = \frac{D^{12T}(T_3)}{D^{12T}(T_1)} = 1\,, \qquad R'_8 = \ell_{\Delta^{12T}}(T_8) = \frac{D^{12T}(T_8)}{D^{12T}(T_1)} = 1\,. \tag{4.10}$$

The R-charges can also equivalently be calculated directly by realizing that the corresponding plane is described by $x + y + 2z = 1$ and using this to see that the erstwhile tachyons now lie *on* this plane.

On the other hand, choosing a different sequence of tachyons by which to fully subdivide the cone gives rise to different fans of subcones. For instance, the dotted line in figure 4 shows a subdivision corresponding to the sequence T_3, T_8, T_1. From the combinations $\frac{1}{3}(\alpha_3 + T_3) = (1, 0, 0)$, $\frac{1}{6}(13T_3 - 2\alpha_3 - 3\alpha_1) = (0, 1, 0)$ and $\frac{1}{2}(-3\alpha_1 - 6\alpha_2 + 13T_3) = (0, 0, 1)$ of the lattice vectors, we see that the subdivision by T_3 gives the subcones $C(0; \alpha_2, \alpha_3, T_3)$, $C(0; T_3, \alpha_3, \alpha_1)$, $C(0; \alpha_1, \alpha_2, T_3)$, to be $\mathbb{Z}_3(0, 1, 1)$, $\mathbb{Z}_6(13, -2, -3) \equiv \mathbb{Z}_6(1, -2, -3)$ and $\mathbb{Z}_2(-3, -6, 13) \equiv \mathbb{Z}_2(1, 0, 1)$ singularities respectively. Further it is easy to see that $T_1 =$

$\frac{1}{3}(T_3 + \alpha_3)$ and $T_8 = \frac{1}{2}(T_3 + \alpha_1)$: thus T_8 becomes marginal after the blowup by T_3 while T_1 acquires the R-charge $R_1' = \frac{2}{3} > \frac{8}{13}$. One can now subdivide by the remaining relevant tachyons to obtain the total volume of the subcones to be $6(1)+3$ (from the list of volumes above): this is greater than the corresponding total volume in the most-relevant-tachyon subdivision, as we saw in general earlier.

5. Type-II theories and all-ring terminality

As in the type-0 theory (see section 2), there are eight rings of operators (anti-)chiral under each of the three copies of the $\mathcal{N}$=2 superconformal algebra, in four conjugate pairs. The chiral GSO projection for type-II $\mathbb{C}^d/\mathbb{Z}_N$ theories preserves spacetime fermions and eliminates the bulk tachyon of the untwisted sector: further it acts nontrivially on these twist fields representing tachyons localized to the singular orbifold subspace, projecting out some of them. We will now describe the structure of the various rings in type-II theories in more detail. Unlike the supersymmetric $\mathbb{C}^3/\mathbb{Z}_{N(p,-p-1)}$ orbifolds where all the $R_j = 1$ twisted states in the (c_X, c_Y, c_Z) ring, which comprise all the blowup modes of the singularity, are preserved and operators in the other rings are projected out, the chiral GSO projection for nonsupersymmetric orbifolds retains some localized tachyons in each ring, as we describe in greater detail below.

We recall (appendix A) that for an orbifold $\mathbb{C}^3/\mathbb{Z}_N$ (k_1, k_2, k_3) to allow a type-II chiral GSO projection admitting spacetime fermions, we must have $\sum_i k_i = even$. Localized closed string tachyons arise in those twisted sectors for which the action on the twist field operators is given by

$$X_j^r \to X_j^r\,(-1)^{E_j^r}\,, \tag{5.1}$$

where the GSO exponent E_j^r depends nontrivially on the twist sector j as well as the specific (anti-)chiral ring r that the twist field belongs to. From appendix A (see e.g. eqs. (A.26) (A.27)), the GSO exponents in type-II theories for twist field operators in the various rings of a $\mathbb{C}^3/\mathbb{Z}_N$ (k_1, k_2, k_3) orbifold are

$$\begin{aligned} E_j &= \sum_i \left[\frac{jk_i}{N}\right], \qquad j = 1, \ldots, N-1 \\ &= \text{odd}\,, \qquad X_j \in (c_X, c_Y, c_Z), (c_X, a_Y, a_Z), (a_X, c_Y, a_Z), (a_X, a_Y, c_Z) \\ &= \text{even}\,, \qquad X_j \in (c_X, c_Y, a_Z), (c_X, a_Y, c_Z), (a_X, c_Y, c_Z), (a_X, a_Y, a_Z)\,. \end{aligned} \tag{5.2}$$

For example, a (c_X, c_Y, c_Z)-ring twist field operator (in the $(-1,-1)$ picture)

$$X_j = \prod_{i=1}^{3} X^i_{\{jk_i/N\}} = \prod_{i=1}^{3} \sigma_{\{jk_i/N\}}\, e^{i\{jk_i/N\}(H-\bar{H})}\,, \tag{5.3}$$

has its conjugate field

$$X_j^* = \prod_{i=1}^{3} (X^i_{\{jk_i/N\}})^* = \prod_{i=1}^{3} (X^i_{\{-(N-j)k_i/N\}})^* = \prod_{i=1}^{3} (X^i_{1-\{(N-j)k_i/N\}})^*\,, \tag{5.4}$$

which[15] clearly lies in the $(N-j)$-th twist sector in the conjugate ring (a_X, a_Y, a_Z). Now it is straightforward to see that if X_j is preserved, so is its conjugate field in the conjugate ring:

$$
\begin{aligned}
E_j &= \sum_i \left[\frac{jk_i}{N}\right] = \text{odd} \Rightarrow \\
E_{N-j} &= \sum_i \left[\frac{(N-j)k_i}{N}\right] = \sum_i k_i + \sum_i \left[\frac{-jk_i}{N}\right] = -E_j - 3 = \text{even}\,.
\end{aligned}
\tag{5.5}
$$

Similarly for the other rings and their conjugates.

As we have briefly mentioned in section 2, there is a convenient notation that can be used to study and label twist operators in the various rings. To illustrate this, note that twist operators in the (c_X, c_Y, a_Z)-ring can be rewritten as

$$
X_j^{cca} = X^1_{\{jk_1/N\}}\, X^2_{\{jk_2/N\}}\, (X^3_{1-\{jk_3/N\}})^* = X^1_{\{jk_1/N\}}\, X^2_{\{jk_2/N\}}\, (X^3_{\{-jk_3/N\}})^*\,, \tag{5.6}
$$

which resemble twist operators in the (c_X, c_Y, c_Z) ring of the orbifold $\mathbb{C}^3/\mathbb{Z}_N\,(k_1, k_2, -k_3)$ with $X^3 \to (X^3)^*$: indeed the R-charges of the operators are identical while the condition on their GSO exponents

$$
E_j = \sum_i \left[\frac{jk_i}{N}\right] = \text{even} \tag{5.7}
$$

can be re-expressed as

$$
\begin{aligned}
E_j^{cca} &= \left[\frac{jk_1}{N}\right] + \left[\frac{jk_2}{N}\right] + \left[-\frac{jk_3}{N}\right] \\
&= E_j + 1 + \text{even} = \text{odd}\,,
\end{aligned}
\tag{5.8}
$$

so that as expected for a (c_X, c_Y, c_Z) ring operator, the corresponding GSO exponent E_j^{cca} is odd. As another example, the (a_X, a_Y, c_Z) ring of $\mathbb{C}^3/\mathbb{Z}_N\,(k_1, k_2, k_3)$ can be expressed as the (c_X, c_Y, c_Z) ring of the orbifold $\mathbb{C}^3/\mathbb{Z}_N\,(-k_1, -k_2, k_3)$ with $X^1 \to (X^1)^*, X^2 \to (X^2)^*$. Generalizing, we see that operators in non-(c_X, c_Y, c_Z) rings of the $\mathbb{C}^3/\mathbb{Z}_N\ (k_1, k_2, k_3)$ orbifold can be expressed as (c_X, c_Y, c_Z) ring operators of a corresponding orbifold with related weights. This interrelation between the structure of the various rings can be exploited to rewrite the GSO exponents for preserved twist operators in the various rings more conveniently, so that they are uniformly *odd* as expected for the re-expressed (c_X, c_Y, c_Z) ring operators. We will find this rewriting of the GSO exponents particularly convenient to use when we discuss all-ring terminality (see e.g. eq. (5.21)).

5.1 The GSO projection for subcones and residual tachyons within

We now come to the question of studying the geometry of subcones in greater detail, in particular in the light of the chiral GSO projection for type-II theories.[16] Consider

[15] We recall that $\{x\} = x - [x]$ denotes the fractional part of x, with $[x]$ the integer part of x (the greatest integer $\leq x$). By definition, $0 \leq \{x\} < 1$. Note that, for $m, n > 0$, we have $[\frac{-m}{n}] = -[\frac{m}{n}] - 1$ and therefore $\{\frac{-m}{n}\} = -\frac{m}{n} - [\frac{-m}{n}] = 1 - \{\frac{m}{n}\}$.

[16] The question of the type-II GSO projection for residual singularities has been studied in the codimension two case in e.g. [13, 14] via the Landau-Ginzburg description of [4].

condensing a GSO-preserved twisted state $T_j = (j, -[\frac{jp}{N}], -[\frac{jq}{N}]) \equiv (\frac{j}{N}, \{\frac{jp}{N}\}, \{\frac{jq}{N}\})$ in a type-II $\mathbb{C}^3/\mathbb{Z}_{N(p,q)}$ orbifold. Then the chiral GSO projection requires that $p+q = odd$ and $[\frac{jp}{N}]+[\frac{jq}{N}] = odd$. This gives three subcones $C(0;T_j,\alpha_2,\alpha_3), C(0;T_j,\alpha_1,\alpha_2), C(0;T_j,\alpha_3,\alpha_1)$, which are orbifolds of the general form $\mathbb{C}^3/\mathbb{Z}_n\,(r',p',q')$. In general these are not isolated. As we have seen from (4.4), $C(0;T_j,\alpha_2,\alpha_3)$ is equivalent to the orbifold $\mathbb{C}^3/\mathbb{Z}_j\,(1,[\frac{jp}{N}],[\frac{jq}{N}])$, the orbifold action being on the coordinates represented by T_j,α_2,α_3 respectively. This clearly admits a type-II GSO projection since $r'+p'+q' = 1+[\frac{jp}{N}]+[\frac{jq}{N}] = even$. The other subcones are somewhat harder to nail down in general but in large classes of cases it is possible to glean insight using the toric data.

Consider the subcone $C(0;T_j,\alpha_3,\alpha_1)$: using the Smith normal form or otherwise, we see that the vector

$$\begin{aligned}(1,0,0) &= \frac{1}{N\{\frac{jp}{N}\}}\left(p\left(T_j + \left[\frac{jq}{N}\right]\alpha_3\right) - \left[\frac{jp}{N}\right](\alpha_1 + q\alpha_3)\right)\\ &= \frac{1}{N\{\frac{jp}{N}\}}\left(pT_j - \left[\frac{jp}{N}\right]\alpha_1 + \left(p\left[\frac{jq}{N}\right] - q\left[\frac{jp}{N}\right]\right)\alpha_3)\right)\end{aligned} \tag{5.9}$$

is in the original lattice, showing that the subcone $C(0;T_j,\alpha_3,\alpha_1)$ is equivalent to the orbifold

$$\mathbb{C}^3/\mathbb{Z}_{N\{\frac{jp}{N}\}}\left(p, -\left[\frac{jp}{N}\right], p\left[\frac{jq}{N}\right] - q\left[\frac{jp}{N}\right]\right),$$

the orbifold action being on the coordinates represented by T_j,α_1,α_3 respectively. It is important to note that such a linear combination of lattice vectors T_j,α_1,α_3 giving a vector in the original lattice is only defined up to adding integer multiples of the lattice vectors. Furthermore this vector is degenerate when any of the coefficients of T_j,α_1,α_3 vanishes: in this case, the subcone is non-isolated. Similarly, the existence and non-degeneracy of the vector

$$(1,0,0) = \frac{1}{N\{\frac{jq}{N}\}}\left(qT_j - \left[\frac{jq}{N}\right]\alpha_1 + \left(q\left[\frac{jp}{N}\right] - p\left[\frac{jq}{N}\right]\right)\alpha_2)\right) \tag{5.10}$$

in the lattice shows that the subcone $C(0;T_j,\alpha_1,\alpha_2)$ is equivalent to the orbifold

$$\mathbb{C}^3/\mathbb{Z}_{N\{\frac{jq}{N}\}}\left(q, -\left[\frac{jq}{N}\right], q\left[\frac{jp}{N}\right] - p\left[\frac{jq}{N}\right]\right),$$

the orbifold action being on the coordinates represented by T_j,α_1,α_2 respectively.

It is interesting to note, assuming validity of the Smith vectors (5.9) (5.10), that these residual orbifold singularities admit a type-II GSO projection: e.g. using the linear combination (5.9) for the subcone $C(0;T_j,\alpha_3,\alpha_1)$, we have

$$\begin{aligned}r'+p'+q' = p - \left[\frac{jp}{N}\right] + p\left[\frac{jq}{N}\right] - q\left[\frac{jp}{N}\right] &= p\left(1 + \text{odd} - \left[\frac{jp}{N}\right]\right) - \left[\frac{jp}{N}\right] - q\left[\frac{jp}{N}\right]\\ &= -\left[\frac{jp}{N}\right](1+p+q) + \text{even} = \text{even}\,.\end{aligned} \tag{5.11}$$

It is important to note that if the order $N\{\frac{jp}{N}\}$ of the discrete group of the subcone is odd, then a unit multiple of each of the lattice vectors T_j,α_3,α_1 can be added to the vector (5.9):

in effect, this has shifted the orbifold weights by $N\{\frac{jp}{N}\}$, which changes the GSO projection to type 0 if $N\{\frac{jp}{N}\}$ = odd. Similarly, we can use the linear combination (5.10) for the subcone $C(0;T_j,\alpha_1,\alpha_2))$ to show that this subcone also admits a type-II projection.

Now let us study subsequent twisted states lying in these subcones and the GSO projection for them. Consider a sub-twisted state $T_k = (k, -[\frac{kp}{N}], -[\frac{kq}{N}]) \equiv (\frac{k}{N}, \{\frac{kp}{N}\}, \{\frac{kq}{N}\})$ that was preserved by the original chiral GSO projection that was performed, i.e. $[\frac{kp}{N}] + [\frac{kq}{N}] = odd$. We can find the renormalized R-charge of T_k w.r.t. the R-symmetries of the subcone that it now lies in, as follows (see also the corresponding subsection in the previous section). If e.g. $T_k \in C(0;T_j,\alpha_2,\alpha_3)$, we can write T_k as a linear combination of the new lattice basis vectors

$$\begin{aligned} T_k &= aT_j + b\alpha_2 + c\alpha_3\,, \\ \text{i.e.} \qquad \left(k, -\left[\frac{kp}{N}\right], -\left[\frac{kq}{N}\right]\right) &= a\left(j, -\left[\frac{jp}{N}\right], -\left[\frac{jq}{N}\right]\right) + b(0,1,0) + c(0,0,1)\,, \end{aligned} \tag{5.12}$$

which can be solved to give

$$R'_k \equiv (a,b,c) = \left(\frac{k}{j},\ \frac{k}{j}\left[\frac{jp}{N}\right] - \left[\frac{kp}{N}\right],\ \frac{k}{j}\left[\frac{jq}{N}\right] - \left[\frac{kq}{N}\right]\right). \tag{5.13}$$

Similarly, if $T_k \in C(0;T_j,\alpha_3,\alpha_1)$, we can write

$$\begin{aligned} T_k &= aT_j + b\alpha_1 + c\alpha_3\,, \\ \text{i.e.} \qquad \left(k, -\left[\frac{kp}{N}\right], -\left[\frac{kq}{N}\right]\right) &= a\left(j, -\left[\frac{jp}{N}\right], -\left[\frac{jq}{N}\right]\right) + b(N,-p,-q) + c(0,0,1)\,, \end{aligned} \tag{5.14}$$

to get

$$R'_k \equiv (a,b,c) = \left(\frac{\{\frac{kp}{N}\}}{\{\frac{jp}{N}\}},\ \frac{k}{N} - \frac{j}{N}\frac{\{\frac{kp}{N}\}}{\{\frac{jp}{N}\}},\ \left\{\frac{kq}{N}\right\} - \left\{\frac{jq}{N}\right\}\frac{\{\frac{kp}{N}\}}{\{\frac{jp}{N}\}}\right). \tag{5.15}$$

Likewise if $T_k \in C(0;T_j,\alpha_1,\alpha_2)$, we can write

$$\begin{aligned} T_k &= aT_j + b\alpha_1 + c\alpha_2\,, \\ \text{i.e.} \qquad \left(k, -\left[\frac{kp}{N}\right], -\left[\frac{kq}{N}\right]\right) &= a\left(j, -\left[\frac{jp}{N}\right], -\left[\frac{jq}{N}\right]\right) + b(N,-p,-q) + c(0,1,0)\,, \end{aligned} \tag{5.16}$$

to get (essentially $p \leftrightarrow q$)

$$R'_k \equiv (a,b,c) = \left(\frac{\{\frac{kq}{N}\}}{\{\frac{jq}{N}\}},\ \frac{k}{N} - \frac{j}{N}\frac{\{\frac{kq}{N}\}}{\{\frac{jq}{N}\}},\ \left\{\frac{kp}{N}\right\} - \left\{\frac{jp}{N}\right\}\frac{\{\frac{kq}{N}\}}{\{\frac{jq}{N}\}}\right). \tag{5.17}$$

In all of the above cases, if $0 < a,b,c < 1$, then T_k is an interior lattice point in the subcone in question. Furthermore if $a+b+c < 1$, then T_k is tachyonic.

Let us now study the exponent required for the GSO w.r.t. the sub-twisted state T_k. From appendix A we recall that for a twisted state T_k with R-charge $R_k = (r_1,r_2,r_3)$ in a $\mathbb{C}^3/\mathbb{Z}_n$ (k_1,k_2,k_3) orbifold, the type-II GSO exponent (A.5) in question can be written in terms of the H-shifts $a_i = odd$ integers as

$$E = \sum_i a_i r_i\,, \qquad \text{with } a_i = \text{odd, satisfying } \sum_i a_i k_i = 0 \pmod{2n}\,. \tag{5.18}$$

Since the original cone is a type-II orbifold singularity, we have $p+q=$ odd: furthermore, since T_j is preserved, $[\frac{jp}{N}]+[\frac{jq}{N}]=$ odd. Now consider $T_k \in C(0;T_j,\alpha_2,\alpha_3) \equiv \mathbb{C}^3/\mathbb{Z}_j\,(1,[\frac{jp}{N}],[\frac{jq}{N}])$. Then we require $a_i=$ odd satisfying $a_1+a_2[\frac{jp}{N}]+a_3[\frac{jq}{N}]=0 \pmod{2j}$: this is solved by $a_1=[\frac{jp}{N}]+[\frac{jq}{N}]$, $a_2=a_3=-1$, each of which is odd. Then we have

$$E=\left(\left[\frac{jp}{N}\right]+\left[\frac{jq}{N}\right]\right)\frac{k}{j}-(1)\left(\frac{k}{j}\left[\frac{jp}{N}\right]-\left[\frac{kp}{N}\right]\right)-(1)\left(\frac{k}{j}\left[\frac{jq}{N}\right]-\left[\frac{kq}{N}\right]\right)=\left[\frac{kp}{N}\right]+\left[\frac{kq}{N}\right], \tag{5.19}$$

which is the same as the original GSO exponent for the state T_k. Thus a sub-twisted state that was preserved by the original GSO, i.e. $E=[\frac{kp}{N}]+[\frac{kq}{N}]=$ odd remains preserved after the subdivision by T_j. Similarly if $T_k \in C(0;T_j,\alpha_3,\alpha_1) \equiv \mathbb{C}^3/\mathbb{Z}_{N\{\frac{jp}{N}\}}\,(p,-[\frac{jp}{N}],p[\frac{jq}{N}]-q[\frac{jp}{N}])$, we require $a_i=$ odd satisfying $a_1p-a_2[\frac{jp}{N}]+a_3p[\frac{jq}{N}]-a_3q[\frac{jp}{N}]=0 \pmod{2N\{\frac{jp}{N}\}}$: this is solved by $a_1=[\frac{jp}{N}]-[\frac{jq}{N}], a_2=p-q, a_3=1$, each of which is odd. Then we have

$$E=\left(\left[\frac{kp}{N}\right]+\left[\frac{kq}{N}\right]\right)+\text{even}\,, \tag{5.20}$$

and likewise for the case $T_k \in C(0;T_j,\alpha_1,\alpha_2) \equiv \mathbb{C}^3/\mathbb{Z}_{N\{\frac{jq}{N}\}}\,(q,-[\frac{jq}{N}],q[\frac{jp}{N}]-p[\frac{jq}{N}])$.

Thus this proves that originally preserved sub-twisted states continue to be preserved after condensation of a preserved twisted state for each of the three subcones. It is important to note however that our proof really only holds for the cases where the subcone in question is in fact the orbifold determined by Smith normal form as above. Indeed as we will see in the examples to follow, condensation of a tachyon in an isolated orbifold does give rise to residual non-isolated singularities with some sub-twisted states lying on a wall of one or more of the subcones. In such cases, the above expression of the Smith normal form that we have used could potentially break down, although in specific examples, it is straightforward to analyze the singularity directly. On this note, it is important to realize that an orbifold $\mathbb{C}^3/\mathbb{Z}_N\,(k_1,k_2,k_3)$ can be written in canonical form $\mathbb{C}^3/\mathbb{Z}_N\,(1,p,q)$, only if at least one of the k_i is relatively prime to N, i.e. $\gcd(k_i,N)=1$ for some k_i, in which case the orbifold has a chance of being isolated. As an example, the non-isolated orbifold $\mathbb{C}^3/\mathbb{Z}_6\,(2,2,3)$, which cannot be expressed in canonical form, cannot be a type-II orbifold since $\sum_i k_i=7=$ odd. Although the proof above does not shed general light on whether an orbifold such as this can arise under condensation of a localized tachyon in a type-II orbifold, this seems unlikely: in particular, we have not found any type-II example where a residual singularity does not admit any type-II GSO projection.

5.2 All-ring terminality

Given the structure of the chiral GSO for the various rings that we have described above, it is interesting to ask if the chiral GSO projection allows geometric terminal singularities to exist as physically sensible type-II theories that admit spacetime fermions projecting out the bulk tachyon. As we have seen, the orbifold $\mathbb{C}^3/\mathbb{Z}_N\,(1,p,N-p)$ with $N=$ odd and p,N coprime is a geometric terminal singularity with all twisted states in the (c_X,c_Y,c_Z) ring being irrelevant (see the example in section 2): this can also admit propagation of type-II strings, since $\sum k_i=1+N=$ *even* implies that the bulk tachyon can be GSO projected out.

However, this does not preclude the existence of GSO-preserved tachyonic twisted states in the other rings. Indeed while these will not be chiral deformations of the theory, they are nonetheless blowup modes which metrically smooth out the singularity. Thus in order to study the physical existence of terminal singularities, we must look for preserved relevant operators arising from any of the various rings.

To study all-ring terminality, it is sufficient for convenience to label distinct orbifolds of the form $\mathbb{Z}_N\ (1,p,q)$ by restricting to the case $p,q>0$. In this notation, the (c_X,c_Y,c_Z) ring of the familiar supersymmetric orbifold is recognized as that ring of the orbifold $\mathbb{C}^3/\mathbb{Z}_N\ (1,p,q)$ such that $1+p-q=0\ (\mathrm{mod}\ N)$, or $1-p+q=0\ (\mathrm{mod}\ N)$.

Then for the $j=1$ sector in some ring to be preserved, we require the corresponding GSO exponent

$$\begin{aligned} E_{ccc} &= \left[\frac{jp}{N}\right]+\left[\frac{jq}{N}\right]=0\,, & E_{cca} &= \left[\frac{jp}{N}\right]+\left[-\frac{jq}{N}\right]=-1\,,\\ E_{cac} &= \left[-\frac{jp}{N}\right]+\left[\frac{jq}{N}\right]=-1\,, & E_{caa} &= \left[-\frac{jp}{N}\right]+\left[-\frac{jq}{N}\right]=-2\,, \end{aligned} \tag{5.21}$$

to be odd. From the above, we see that the $j=1$ state survives in the (c_X,c_Y,a_Z) and (c_X,a_Y,c_Z) rings. These have the R-charges $(\frac{1}{N},\frac{p}{N},1-\frac{q}{N})$ and $(\frac{1}{N},1-\frac{p}{N},\frac{q}{N})$. For the orbifold to be "string-terminal", it is necessary (but not sufficient) to have these surviving $j=1$ states be irrelevant:

$$\frac{1}{N}+\frac{p}{N}+1-\frac{q}{N}>1\,,\qquad \frac{1}{N}+1-\frac{p}{N}+\frac{q}{N}>1\,, \tag{5.22}$$

i.e. we want $1+p>q$, $1+q>p$. This gives $p>q-1$.

Now recall that to admit a type-II GSO projection, we require $p+q=$ odd. This implies that $p\neq q$ (since, as we have mentioned above, including all rings of the theory, it suffices[17] to consider $p,q>0$). We can assume without loss of generality that $p<q$. This gives

$$q-1<p<q\,, \tag{5.23}$$

which has no solution for integer $p,q>0$.

This shows that type-II all-ring terminality is not possible whenever the orbifold action is written in the canonical form $(1,p,q)$: this includes all isolated orbifolds. However, as we have mentioned in the previous subsection, the orbifold action (k_1,k_2,k_3) for non-isolated singularities (as before, restrict to $k_i>0$, since all rings are being considered) can be cast in the form $(1,p,q)$, only if at least one of the k_i is relatively prime to N, i.e. there exists some l such that $\{\frac{lk_i}{N}\}=1$ for some k_i. Note however that for a strongly non-isolated singularity, there is no k_i coprime with N, i.e. there always exist some l_i such that $\{\frac{l_ik_i}{N}\}=0$, for $i=1,2,3$: such twist sectors are likely to be the minimum R-charge sectors, i.e. the sectors where some ring is likely to have a tachyon. Therefore, without loss of generality, assume $\{\frac{l_1k_1}{N}\}=0$. Then the R-charges for the corresponding states from the

[17]Note that, e.g. shifting one of the orbifold weights by $N=$ odd, is already taken care of by including all rings.

(c_X, c_Y, c_Z), (c_X, c_Y, a_Z), (c_X, a_Y, c_Z), (c_X, a_Y, a_Z) rings are

$$\begin{aligned} R_l^{ccc} &= \left\{\frac{lk_2}{N}\right\} + \left\{\frac{lk_3}{N}\right\}, \\ R_l^{cca} &= \left\{\frac{lk_2}{N}\right\} + 1 - \left\{\frac{lk_3}{N}\right\}, \\ R_l^{cac} &= 1 - \left\{\frac{lk_2}{N}\right\} + \left\{\frac{lk_3}{N}\right\}, \\ R_l^{caa} &= -\left\{\frac{lk_2}{N}\right\} - \left\{\frac{lk_3}{N}\right\} + 2\,, \end{aligned} \tag{5.24}$$

with the corresponding GSO exponents being E_l for the $(c_X, c_Y, c_Z), (c_X, a_Y, a_Z)$ states and $E_l + 1$ for the $(c_X, c_Y, a_Z), (c_X, a_Y, c_Z)$ states (upto even integers, which do not affect the sign of the exponent). Notice that these look like the corresponding expressions for codimension two. Now if $E_l = even$, the $(c_X, c_Y, c_Z), (c_X, a_Y, a_Z)$ states are projected out, and we require for string-terminality that the GSO-surviving states are irrelevant, in other words, $R_l^{cca}, R_l^{cac} > 1$, i.e.

$$\left\{\frac{lk_2}{N}\right\} > \left\{\frac{lk_3}{N}\right\}, \qquad \left\{\frac{lk_3}{N}\right\} > \left\{\frac{lk_2}{N}\right\}, \tag{5.25}$$

which strict inequality has no solution. Similarly, if $E_l =$ odd, then we require for string-terminality that $R_l^{ccc}, R_l^{caa} > 1$: again it is straightforward to show the absence of any solution. Similarly if $\{\frac{l_i k_i}{N}\} = 0$, for $i = 2, 3$. Equality in these expressions indicates the appearance of marginal states, which can of course resolve the singularity. This shows that there are always tachyonic or marginal operators that arise in such sectors.

This negative result for string-terminality shows that the endpoints of condensation of all tachyons in type-II nonsupersymmetric unstable $\mathbb{C}^3/\mathbb{Z}_N$ orbifolds are always smooth spaces. In other words, type-II string propagation in four noncompact dimensions always resolves potential orbifold terminal singularities.

Along these lines, it is also interesting to ask if there are all-ring terminal $\mathbb{C}^3/\mathbb{Z}_N\,(1, p, q)$ singularities in the type-0 theory. In this case, all twisted states in all rings are preserved by the diagonal GSO projection. Then consider the $j = 1$ sector as before. It is necessary that the $j = 1$ states be irrelevant for terminality. We therefore require that the R-charges for the corresponding states in the various rings satisfy

$$\begin{aligned} \frac{1}{N} + \frac{p}{N} + \frac{q}{N} > 1\,, \qquad & \frac{1}{N} + \frac{p}{N} + 1 - \frac{q}{N} > 1\,, \\ \frac{1}{N} + 1 - \frac{p}{N} + \frac{q}{N} > 1\,, \qquad & \frac{1}{N} + 1 - \frac{p}{N} + 1 - \frac{q}{N} > 1\,. \end{aligned} \tag{5.26}$$

These simplify to give

$$N - 1 < p + q < N + 1\,, \qquad 1 + p > q\,, \qquad 1 + q > p\,. \tag{5.27}$$

Now if $p < q$, then as before we have $q - 1 < p < q$, which is not possible for integer $p, q > 0$. Therefore consider $q = p$. Then we have $N - 1 < 2p < N + 1$. If $N =$ odd $= 2l + 1$, we have $2l < 2p < 2l+2$, which is not possible for integer $p, l > 0$. If $N = \text{even} = 2l$, we have $2l-1 < 2p < 2l+1$, which gives $p = l = \frac{N}{2}$. This is the orbifold $\mathbb{C}^3/\mathbb{Z}_N\,(1, \frac{N}{2}, \frac{N}{2})$. In this case however, the $j=2$ twisted state, if it exists, has R-charges $(\frac{2}{N}, 0, 0)$, which is irrelevant only if $N < 2$: this however does not give any nontrivial orbifold. Therefore the $j=2$ state does not exist, i.e. $N=2$. Thus the only isolated all-ring terminal singularity is $\mathbb{C}^3/\mathbb{Z}_2\,(1,1,1)$: the only twisted states, coming from the $j=1$ sector, are irrelevant in all rings since the corresponding R-charges satisfy

$$\begin{aligned}
(c_X, c_Y, c_Z):&\quad \frac{1}{2}+\frac{1}{2}+\frac{1}{2} > 1\,,\\
(c_X, c_Y, a_Z):&\quad \frac{1}{2}+\frac{1}{2}+1-\frac{1}{2} > 1\,,\\
(c_X, a_Y, c_Z):&\quad \frac{1}{2}+1-\frac{1}{2}+\frac{1}{2} > 1\,,\\
(c_X, a_Y, a_Z):&\quad \frac{1}{2}+1-\frac{1}{2}+1-\frac{1}{2} > 1\,.
\end{aligned} \tag{5.28}$$

Note that $\mathbb{C}^3/\mathbb{Z}_2\,(1,1,1)$ does not admit a chiral GSO projection since $\sum k_i = odd$: thus it does not admit propagation of type-II strings.

Given all this, we expect the following: physically in a given unstable (UV) orbifold, the most relevant tachyon(s) will belong to one (or more) of the (anti-)chiral rings. When this condenses, it creates an expanding bubble of flat space blowing up the divisor it corresponds to. Metrically we expect that this most relevant tachyon pulse will trigger condensation of all tachyons within the same ring, since they preserve the same fraction of the original $\mathcal{N}=(2,2)$ supersymmetry. The endpoint of condensation of all tachyons within this ring will in general include *geometric* terminal singularities. However as the nonexistence proof above shows, these will contain further *nonchiral* blowup modes which will metrically smooth out the corresponding singularities.

We describe some type-II examples below.

Example $\mathbb{C}^3/\mathbb{Z}_{13}\ (1,2,5)$**.** Recall the type-0 example we discussed earlier (figure 4): the most relevant tachyon $T_1^{ccc} \equiv (\frac{1}{13}, \frac{2}{13}, \frac{5}{13}) = \frac{8}{13}$ belonged to the (c_X, c_Y, c_Z) ring. We saw there that the endpoint of the most relevant tachyon sequence included the all-ring terminal singularity $\mathbb{C}^3/\mathbb{Z}_2\ (1,1,1)$. As we have seen above, this does not admit a type-II GSO projection so that tachyon condensation in this type-0 theory cannot result in a type-II theory.

On the other hand, $\mathbb{C}^3/\mathbb{Z}_{13}\ (1,2,5)$ itself admits a consistent type-II GSO projection. In this case, it is straightforward to see that T_1^{ccc} above is in fact GSO-projected out. Of the GSO-surviving tachyons, there turn out to be *two* distinct tachyons with the same R-charge $R = \frac{9}{13}$: these are $T_2^{cca} \equiv (\frac{2}{13}, \frac{4}{13}, \frac{3}{13})$ and $T_5^{caa} \equiv (\frac{5}{13}, \frac{3}{13}, \frac{1}{13})$, belonging to the (c_X, c_Y, a_Z) and (c_X, a_Y, a_Z) rings respectively. The subdivisions thereof and the endpoints of tachyon condensation for either of them condensing alone are straightforward to work out. On the other hand, the methods we use fail if both condense simultaneously: condensation of such mixed tachyons breaks $\mathcal{N}=(2,2)$ worldsheet supersymmetry.

Example $\mathbb{C}^3/\mathbb{Z}_{23}\,(1,4,-11)$**.** See figure 5. We consider the type-II theory here: the (c_X, c_Y, c_Z) ring tachyons $T_1 = (\frac{1}{23}, \frac{4}{23}, \frac{12}{23})$, $T_2 = (\frac{2}{23}, \frac{8}{23}, \frac{1}{23})$, $T_8 = (\frac{8}{23}, \frac{9}{23}, \frac{4}{23})$ with R-charges $R_1 = \frac{17}{23}$, $R_2 = \frac{11}{23}$, $R_8 = \frac{21}{23}$ respectively survive the chiral GSO projection. While there are GSO-preserved tachyons in the other rings, the most relevant tachyon in this theory in fact is T_1 above, from the (c_X, c_Y, c_Z) ring.[18] The vertices of the affine hyperplane of marginal operators are $\alpha_1 = (23, -4, 11)$, $\alpha_2 = (0, 1, 0)$, $\alpha_3 = (0, 0, 1)$ while the tachyons correspond to the lattice vectors $T_1 = (1, 0, 1)$, $T_2 = (2, 0, 1)$, $T_8 = (8, -1, 4)$. T_1 and T_2 are coplanar with α_3. The volumes of some subcones are

$$\begin{aligned}
&V(\alpha_1,\alpha_2,T_8) = V(\alpha_1,T_2,T_1) = V(\alpha_3,\alpha_1,T_1) = 4, \quad V(\alpha_2,\alpha_3,T_2) = 2, \quad V(\alpha_1,T_2,T_8) = 1,\\
&V(\alpha_3,T_1,T_8) = V(\alpha_2,\alpha_3,T_1) = V(\alpha_2,T_1,T_2) = V(T_1,T_2,T_8) = V(\alpha_1,\alpha_2,T_2) = 1,\\
&V(\alpha_2,T_2,T_8) = 0, \quad V(\alpha_2,\alpha_3,T_8) = V(\alpha_3,\alpha_1,T_2) = 8, \quad V(\alpha_3,\alpha_1,T_8) = 9. \qquad (5.29)
\end{aligned}$$

As before let us analyze the sequence of most relevant tachyons, i.e. T_2, T_1, T_8. Condensation of T_2 gives the residual subcones $C(0;\alpha_1,\alpha_2,T_2)$, $C(0;T_2,\alpha_2,\alpha_3)$, $C(0;T_2,\alpha_3,\alpha_1)$, which correspond to flat space, $\mathbb{Z}_2\,(1,0,-1)$ and $\mathbb{Z}_8\,(23,-1,-2) \equiv \mathbb{Z}_8\,(1,1,2)$ singularities respectively, using the Smith normal form: alternatively this can be seen by realizing the combinations $\frac{1}{2}(T_2 - \alpha_3) = (1,0,0)$ and $\frac{1}{8}(23T_2 - \alpha_3 - 2\alpha_1) = (0,1,0)$ of the lattice vectors (shifting by integer multiples thereof). Since $T_1 = \frac{1}{2}(T_2 + \alpha_3)$, it is clear that T_1 is marginal after condensation of T_2. It is straightforward to work out the twisted states of these residual orbifolds and map T_1 onto the corresponding new twisted sector (it is important however to be careful in finding the correct type-II projection for the residual orbifolds which is consistent with the original theory). On the other hand, the subsequent tachyon T_8 with renormalized R-charge

$$R_8' = \ell_{\Delta^{31T}}(T_8) = \frac{D^{31T}(T_8)}{D^{31T}(T_2)} = \frac{21}{23}\left[1 + \frac{9}{8}\,\frac{1-\frac{11}{23}}{\frac{21}{23}}\right] = \frac{3}{2} > 1 \qquad (5.30)$$

has become irrelevant after T_2 condenses! In fact, we have $T_8 = \frac{9}{8}T_2 + \frac{1}{4}\alpha_1 + \frac{1}{8}\alpha_3$, i.e. one of the coefficients is greater than unity. In figure 5, the solid lines correspond to the sequence of most relevant tachyons, while the lightly shaded lines correspond to the subdivision by the now irrelevant T_8. The total volume of the subcones with this sequence of subdivisions is $V_{\text{total}} = 8 + 2 + 1 = 11$. We can now only subdivide by the remaining now-marginal operator T_1 since T_8 being irrelevant does not affect the conformal field theory. Realizing the lattice vector combinations $\frac{1}{4}(23T_1 - 16\alpha_3 - \alpha_1) = (0,1,0)$ and $\frac{1}{4}(24T_2 - 2T_1 - 2\alpha_1) = (0,1,0)$, we see that the T_1 subdivision results in the subcones $C(0;T_1,\alpha_3,\alpha_1)$ and $C(0;T_2,T_1,\alpha_1)$, which are re-

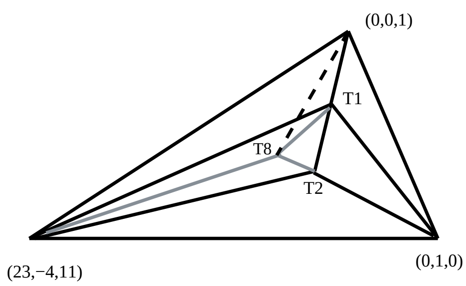

Figure 5: $\mathbb{C}^3/\mathbb{Z}_{23}\,(1,4,-11)$: The three points $\alpha_1, \alpha_2, \alpha_3$ defining the affine hyperplane Δ are shown, along with the five tachyons T_2, T_1, T_8 in type II. The solid lines show the subdivisions corresponding to the sequence of most relevant tachyons while the dotted lines show some possible flips.

[18] One could, if one so wishes, *define* the orbifold action so that the most relevant tachyon lies in the (c_X, c_Y, c_Z) ring.

spectively $\mathbb{Z}_4\,(1,0,1)$ and $\mathbb{Z}_4\,(0,1,1)$ geometric terminal singularities (since the potential tachyons do not survive the type-II GSO projection). However we must realize that both of these are secretly supersymmetric $\mathbb{Z}_4\,(1,-1)$ singularities when twisted states in the other (anti-)chiral rings are taken into account. Thus the final endpoint of the most-relevant-tachyon sequence in this type-II theory includes only flat and supersymmetric spaces.

On the other hand, note that there are flip transitions (shown by the dotted lines) if T_8 condenses first followed by T_2, T_1, landing up at distinct endpoints via condensation of different sequences of tachyons. Since $T_2 = \frac{1}{4}(T_8 + \alpha_2)$, T_2 remains relevant with renormalized R-charge $\frac{1}{2}$ after T_8 has condensed. The subsequent tachyon T_1 has renormalized R-charge

$$R_1' = \ell_{\Delta^{23T}}(T_1) = \frac{D^{23T}(T_1)}{D^{23T}(T_8)} = \frac{17}{23}\left[1 + \frac{1}{8}\,\frac{1-\frac{21}{23}}{\frac{17}{23}}\right] = \frac{3}{4} > \frac{17}{23} \tag{5.31}$$

The total volume of the subcones in this case is $V_{\rm total} = 9 + 6(1) = 15 > 11$, which verifies the fact that the most relevant tachyon sequence gives minimal total volume for the subcones.

6. Conclusions

We have studied condensation of localized tachyons in $\mathbb{C}^3/\mathbb{Z}_N$ nonsupersymmetric orbifolds via the worldsheet RG flows induced thereby. We have seen that this generically leads to a set of decoupled residual geometries that include *geometric* terminal singularities, with no marginal or relevant Kähler blowups by which they can be resolved (although generic metric blowup modes generically do exist). Treated as geometric spaces, they thus admit no canonical resolution and the various possible distinct resolutions via condensation of distinct sequences of tachyons are sometimes related by flip transitions. In general, the renormalized R-charges of subsequent tachyons in the residual geometries are higher than their previous values. Thus the residual geometries in general are more prone to becoming terminal singularities after tachyon condensation. For type-II theories with no bulk tachyon, we have shown that all-ring terminal singularities cannot exist, which shows that the endpoint of tachyon condensation in type-II unstable $\mathbb{C}^3/\mathbb{Z}_N$ orbifold theories are always smooth spaces.

The calculations via toric geometry described in this paper are essentially a reflection of the physics underlying gauged linear sigma models. In particular, topological twisted GLSMs may be reliably used to study tachyon condensation not simply at the endpoints of but *all along* the worldsheet RG flow and map out the phase structure of two dimensional theories including tachyons.

The methods we have used here are of course not powerful enough to study situations where, for instance, mixed tachyons (combinations of tachyons from distinct rings) condense simultaneously. In such cases, we lose control over the system because $(2,2)$ worldsheet supersymmetry breaks down.

We now make a few brief comments on the physics seen by the worldvolume theory on a D-brane probe of a nonsupersymmetric orbifold. In general, closed string twist fields appear as Fayet-Iliopoulos D-term couplings in the D-brane probe theory [32, 3, 5]. Here,

JHEP08(2004)047

the closed string twist fields that are tachyonic condense in time and thus have a time-dependent expectation value, say of functional form $T(t)$. Via the D-term equations, these induce time-dependent Higgs expectation values for the bifundamental link fields of the quiver, which are then proportional to $\sqrt{T(t)}$. For simplicity, let us assume that closed string tachyon condensation occurs so as to monotonically increase the condensate value $T(t)$. Then the link field expectation values also increase in time. Consider a low energy observer on a D-brane probe who observes physics at energy E. Then the link field vevs increase monotonically in time so that the link fields are naturally integrated out in time from the point of view of the low energy observer, thereby leaving a residual quiver with fewer link fields and a less singular [3] orbifold.[19]

It would be interesting to analyze the worldvolume D-brane gauge theories on $\mathbb{C}^3/\mathbb{Z}_N$ orbifold singularities and study their implications, in part with a view to constructing stable nonsupersymmetric string vacua.

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A. The GSO projection

Our discussion of the chiral GSO is based on and generalizes that appearing in [11, 12, 3, 4, 5, 2, 13]. We first outline a method to engineer the GSO projection based essentially on its action on the untwisted sector and consistency thereof with supersymmetric orbifolds. Then we work out the RNS partition function and, by requiring modular invariance thereof, obtain the GSO projection.

A.1 "Engineering" the GSO projection

Here we outline a method to engineer the chiral GSO projection for a type II orbifold $\mathbb{C}^3/\mathbb{Z}_N$ (k_1, k_2, k_3) with k_i not necessarily equal to one. We complexify the eight transverse

[19] A (gauge-fixed) block-spin-like transformation that coarse-grains matrix representations of various D-brane configurations was studied in [33]. In particular [34] studied (in part along similar lines) a block-spin-like transformation on a simplified subset of quiver gauge theories that arise on the worldvolumes of D-brane probes of supersymmetric orbifolds by sequentially Higgsing the gauge symmetry using the bifundamental scalar link fields present in these theories. From this point of view, the image branes for a nonsupersymmetric orbifold naturally form "block-(image)branes" in time in the process of condensation of a localized tachyon. For instance, as the link X_{ij} is integrated out below energies E, the images i and j form the block-(image)brane ij. The "upstairs" matrices of the image branes do not coarse-grain in the homogeneous fashion studied in [33, 34]. Instead row i and column j are deleted from the $N \times N$ matrix to get the $(N-1) \times (N-1)$ "upstairs" matrices of the residual orbifold.

untwisted fermions into $\psi_i,\ i = 0, 1, 2, 3$. Consider a symmetry acting on the untwisted (complex) fermions and the twist fields via $H_i \to H_i + a_i\pi$, i.e.

$$\begin{aligned} \psi_i &\to \psi_i e^{i a_i \pi}, \\ X_j &\to X_j \exp\left[i\pi \sum_i a_i \left\{ \frac{j k_i}{N} \right\} \right] \equiv X_j \, (-1)^{E_j} . \end{aligned} \tag{A.1}$$

This defines a $(-1)^{F_L}$ $\mathbb{Z}_2$ action on the untwisted sector thus eliminating the bulk tachyon only if the a_i are odd integers. The action on the twisted states X_j is a well-defined $\mathbb{Z}_2$ if the exponent E_j is an integer. This GSO exponent can be written as

$$E_j = \sum_i a_i \left\{ \frac{j k_i}{N} \right\} = \frac{j}{N} \sum_i a_i k_i - \sum_i a_i \left[\frac{j k_i}{N} \right]. \tag{A.2}$$

Thus E_j is integral if we have $a_i =$ odd satisfying $\sum_i a_i k_i = 0 \,(\text{mod } 2N)$. Consider the case $\sum_i k_i =$ odd. Then

$$\sum a_i k_i = a_1 k_1 + a_2 k_2 + a_3(\text{odd} - k_1 - k_2) = (a_1 - a_3) k_1 + (a_2 - a_3) k_2 + \text{odd} = \text{odd} \tag{A.3}$$

since the first two terms (containing differences of two odd integers) are each even. Thus this shows that no $a_i =$ odd exist satisfying $\sum_i a_i k_i = 0 \,(\text{mod } 2N)$: in other words, for even N and $\sum_i k_i =$ odd, no chiral GSO projection exists (note that for odd N, one can always shift say $k_3 \to k_3 \pm N$ to make $\sum_i k_i$ even and thereby recover a chiral GSO projection).

For $\sum_i k_i = even$, E_j simplifies to

$$\begin{aligned} E_j = \sum_i a_i \left[\frac{j k_i}{N} \right] &= \sum_i \left[\frac{j k_i}{N} \right] + \text{even} \\ &= \frac{j}{N} \sum_i k_i - R_j + \text{even} , \end{aligned} \tag{A.4}$$

where $R_j = \sum_i \{ \frac{j k_i}{N} \}$ is the R-charge of the twisted state. Thus for a twisted state T with R-charge $R = (r_1, r_2, r_3)$ in the orbifold $\mathbb{C}^3/\mathbb{Z}_N$ (k_1, k_2, k_3), the GSO exponent is

$$E = \sum_i a_i r_i, \qquad \text{with } a_i = \text{odd and } \sum a_i k_i = 0 \quad (\text{mod } 2N) . \tag{A.5}$$

Let us examine this GSO exponent in greater detail to elucidate its properties.

For $\mathbb{C}/\mathbb{Z}_N$, consider the action $H \to H + a\pi$, i.e.

$$\psi \to \psi \, (-1)^a , \qquad X_j \to X_j \, (-1)^{ja/N} , \tag{A.6}$$

which defines a nontrivial $(-1)^{F_L}$ $\mathbb{Z}_2$ action on the untwisted sector only if a is an odd integer. For odd $N = 2M + 1$ prime and general odd $a = 2b + 1$, the action on the twisted states X_j defines a $\mathbb{Z}_2$ only if $a = N$. For other odd N, there are appropriate twist-j subsectors with prime factors for which the argument is then the same. Indeed let us consider even $N = 2^k$ with $a = 2b + 1$, so that the twisted states transform as

$$X_j \to X_j \, (-1)^{ja/N} = X_j \, (-1)^{j(2b+1)/(2^k)} \tag{A.7}$$

Then for no twist-j subsector is there a well-defined $\mathbb{Z}_2$. For general even N=2M, there are twist-j subsectors which are identical to the above cases. Thus we take $a = N$ with N odd for type-II $\mathbb{C}/\mathbb{Z}_N$. Then the chiral GSO above acts as

$$\psi \to \psi\,(-1)^N\,, \qquad X_j \to X_j\,(-1)^j\,. \tag{A.8}$$

Since the $(-1,-1)$ picture vertex operators are odd under chiral GSO, only the X_j states with odd j survive.

Now consider $\mathbb{C}^2/\mathbb{Z}_{N(p)}$ and $\mathbb{C}^3/\mathbb{Z}_{N(p,q)}$. In these cases, the GSO exponent E_j is an integer if $a_1 + a_2 p + a_3 q = 0(\text{mod 2N})$.

From above, we have $1 + p + q = even$. For $\mathbb{C}^2/\mathbb{Z}_{N(p)}$, we have $q = 0$: then $p =$ odd and $a_1 = p, a_2 = -1$ satisfy $a_1 + a_2 p = 0\,(\text{mod}2N)$. We have then the chiral GSO action

$$\psi_1 \to \psi_1\ (-1)^p\,, \qquad \psi_2 \to \psi_2\ (-1)\,, \qquad X_j \to X_j\ (-1)^{[jp/N]} \tag{A.9}$$

for the $\mathbb{C}^2/\mathbb{Z}_{N(p)}$ orbifold.

Note that the supersymmetric orbifold is $p = -1$ in this convention, so that $[\frac{jp}{N}] = [-\frac{j}{N}] = -1$. Thus the chiral GSO acts as $X_j \to X_j(-1)$ for all twisted states, which in fact are marginal with R-charge $R_j = \frac{j}{N} + \{-\frac{j}{N}\} = 1$. Thus the entire (c_X, c_Y) ring is preserved by the chiral GSO in this case, while the entire (c_X, a_Y) ring is projected out.

For $\mathbb{C}^3/\mathbb{Z}_{N(p,q)}$, we have $p+q =$ odd: then $a_1 = p+q$, $a_2 = a_3 = -1$ satisfy $a_1 + a_2 p + a_3 q = 0\,(\text{mod } 2N)$. Then the GSO exponent E_j is

$$E_j = \left[\frac{jp}{N}\right] + \left[\frac{jq}{N}\right] = \frac{j}{N}(1+p+q) - \frac{j}{N} - \left\{\frac{jp}{N}\right\} - \left\{\frac{jq}{N}\right\} = \frac{j}{N}(1+p+q) - R_j\,. \tag{A.10}$$

and we have the chiral GSO action

$$\psi_1 \to \psi_1\,(-1)^{p+q}\,, \qquad \psi_2 \to \psi_2\,(-1)\,, \qquad \psi_3 \to \psi_3\,(-1)\,, \qquad X_j \to X_j\,(-1)^{[jp/N]+[jq/N]}\,. \tag{A.11}$$

For the supersymmetric case $1+p+q = 0$ and we have $E = -R_j$, so that the chiral GSO acts as $X_j \to X_j(-1)$ for all twisted states in the (c_X, c_Y, c_Z) ring with R-charge $R_j = 1$. The blowup modes of the geometry are defined purely in terms of the marginal $R_j = 1$ states for the supersymmetric case so that they are all preserved by the GSO (note that there also exist $R_j = 2$ twisted states here, unlike in the $\mathbb{C}^2/\mathbb{Z}_{N(p)}$ case). Thus in general, the chiral GSO for type-II theories projects out the twist operators X_j of the (c_X, c_Y, c_Z) ring with $[\frac{jp}{N}] + [\frac{jq}{N}] \in 2\mathbb{Z}$ and retains those with either $[\frac{jp}{N}]$ or $[\frac{jq}{N}]$ odd (but not both). Similarly in the (c_X, c_Y, a_Z) ring, the X_j with $[\frac{jp}{N}] + [\frac{jq}{N}]$ even are retained and so on.

Note that this agrees with the Green-Schwarz GSO analysis (generalized from that in [3]), which starts with a rotation generator

$$R = \exp\left[\frac{2\pi i}{N}(J_{45} + pJ_{67} + qJ_{89})\right] \tag{A.12}$$

Then

$$R^N = (-1)^{2(s_{45}+ps_{67}+qs_{89})} \tag{A.13}$$

so that considering the action thereof on various spinor charge sectors $(\pm 1/2, \pm 1/2, \pm 1/2)$, and demanding $R^N = 1$ for removing the bulk tachyon gives $p + q =$ odd.

It is important to note that changing $q \to q+N$ introduces an extra factor of $(-1)^N$ in (A.11). Also $p+q=$ odd now changes to $p+q+N=$ even thus reversing the bulk tachyon projection and changing a type-II theory to type 0. For example, the $\mathbb{C}^3/\mathbb{Z}_{11(2,-7)}$ orbifold is a good type-II theory with the bulk tachyon projected out, whereas the orbifold $\mathbb{C}^3/\mathbb{Z}_{11(2,4)}$ (which is equivalent in conformal field theory) does have a bulk tachyon and should be regarded as type 0. Furthermore, the action on the twisted states changes since the exponent E becomes $E \to j+E$, so that supersymmetric orbifolds now have $E \to \frac{j}{N}(1+p+q+N) - R_j = j - R_j$, thus projecting down to $j=$ even twisted states among the $R_j = 1$ blowup modes.

A.2 Modular invariance of the partition function

The partition function for $\mathbb{C}/\mathbb{Z}_N$ was described notably in [11, 12], using both the Green-Schwarz and the RNS formulations. We primarily use the RNS formulation here.

The type-0 string has a diagonal GSO projection that ties together the left and right movers: it has the partition function

$$\begin{aligned} Z = {} & \frac{1}{2N}\sum_{j,l=0}^{N-1}\frac{1}{|\eta^2(\tau)\bar{\eta}^2(\bar{\tau})|}\prod_{i=1}^{3}\left|\frac{\eta(\tau)}{\theta\begin{bmatrix}\frac{1}{2}+jk_i/N\\ \frac{1}{2}+lk_i/N\end{bmatrix}(0,\tau)}\right|^2 \times \\ & \times\left[\left|\prod_{i=1}^{3}\theta\begin{bmatrix}\frac{jk_i}{N}\\ \frac{lk_i}{N}\end{bmatrix}\theta\begin{bmatrix}0\\ 0\end{bmatrix}\right|^2 + \left|\prod_{i=1}^{3}\theta\begin{bmatrix}\frac{jk_i}{N}\\ \frac{lk_i}{N}+\frac{1}{2}\end{bmatrix}\theta\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}\right|^2 + \right. \\ & \left. + \left|\prod_{i=1}^{3}\theta\begin{bmatrix}\frac{jk_i}{N}+\frac{1}{2}\\ \frac{lk_i}{N}\end{bmatrix}\theta\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}\right|^2 \pm \left|\prod_{i=1}^{3}\theta\begin{bmatrix}\frac{jk_i}{N}+\frac{1}{2}\\ \frac{lk_i}{N}+\frac{1}{2}\end{bmatrix}\theta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}\right|^2\right]. \end{aligned} \tag{A.14}$$

Clearly this partition function exists for any k_i, N.

On the other hand, the 1-loop partition function on a $\mathbb{C}^3/\mathbb{Z}_N$ (k_1, k_2, k_3) orbifold for a type-II string with separate GSO projections on the left and right movers is given by the sum over twisted sectors as

$$Z = \frac{1}{4N}\sum_{j,l=0}^{N-1}\frac{1}{|\eta^2(\tau)\bar{\eta}^2(\bar{\tau})|}\prod_{i=1}^{3}\left|\frac{\eta(\tau)}{\theta\begin{bmatrix}\frac{1}{2}+jk_i/N\\ \frac{1}{2}+lk_i/N\end{bmatrix}(0,\tau)}\right|^2 \left|\frac{\zeta_l^j}{|\eta^4(\tau)}\right|^2 \tag{A.15}$$

where

$$\begin{aligned} \zeta_l^j = {} & \prod_{i=1}^{3}\theta\begin{bmatrix}\frac{jk_i}{N}\\ \frac{lk_i}{N}\end{bmatrix}\theta\begin{bmatrix}0\\ 0\end{bmatrix} - e^{-i\pi\sum_i\frac{jk_i}{N}}\prod_{i=1}^{3}\theta\begin{bmatrix}\frac{jk_i}{N}\\ \frac{lk_i}{N}+\frac{1}{2}\end{bmatrix}\theta\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix} - \\ & - \prod_{i=1}^{3}\theta\begin{bmatrix}\frac{jk_i}{N}+\frac{1}{2}\\ \frac{lk_i}{N}\end{bmatrix}\theta\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix} - e^{-i\pi\sum_i\frac{jk_i}{N}}\prod_{i=1}^{3}\theta\begin{bmatrix}\frac{jk_i}{N}+\frac{1}{2}\\ \frac{lk_i}{N}+\frac{1}{2}\end{bmatrix}\theta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix} \end{aligned} \tag{A.16}$$

contains the sum over spin structures for the j-th twisted sector twisted by g^l in the "time" direction. The terms in Z are easily recognized as the contributions from the untwisted bosons in the one complex flat dimension, multiplied by the contributions from the twisted bosons in the three orbifolded complex dimensions and the fermionic contributions.

At this point, we list some formulae involving theta functions that we use here (from [11, 35]). The boundary conditions on the worldsheet scalars and spinors in the NS sector are

$$\begin{aligned} X(w+2\pi) &= e^{2\pi i a} X(w)\,, \qquad X(w+2\pi\tau) = e^{2\pi i b} X(w)\,, \\ \psi(w+2\pi) &= -e^{2\pi i a}\psi(w) \equiv -e^{-i\pi\alpha}\psi(w)\,, \\ \psi(w+2\pi\tau) &= -e^{2\pi i b}\psi(w) \equiv -e^{-i\pi\beta}\psi(w)\,. \end{aligned} \tag{A.17}$$

A chiral fermion with hamiltonian

$$H_a = \sum_{n=1}^{\infty}\left[\left(n-\frac{1}{2}+a\right)d_n^\dagger d_n + \left(n-\frac{1}{2}-a\right)\bar{d}_n^\dagger \bar{d}_n\right] + \frac{a^2}{2} - \frac{1}{24} \tag{A.18}$$

has the partition function

$$\begin{aligned} Z_\beta^\alpha &= \mathrm{Tr}(h_b q^{H_a}) = \frac{\theta[{a \atop b}]}{\eta(\tau)} \equiv \frac{\theta\begin{bmatrix} -\frac{\alpha}{2} \\ -\frac{\beta}{2} \end{bmatrix}}{\eta(\tau)} \\ &= e^{2\pi i a b}\, q^{\frac{a^2}{2}-\frac{1}{24}} \prod_{n=1}^{\infty}\left(1+q^{n-\frac{1}{2}+a}\, e^{2\pi i b}\right)\left(1+q^{n-\frac{1}{2}-a}\, e^{-2\pi i b}\right) \end{aligned} \tag{A.19}$$

where the $\mathbb{Z}_N$ action on the Hilbert space is

$$h_b d h_b^{-1} = -e^{-2\pi i b} d\,, \qquad \bar{h}_b \bar{d} \bar{h}_b^{-1} = -e^{2\pi i b}\bar{d}\,. \tag{A.20}$$

Some useful formulae involving theta functions are

$$\begin{aligned} \theta\begin{bmatrix} a \\ b \end{bmatrix}(\nu,\tau) &= e^{i\pi a^2\tau + 2\pi i a(\nu+b)}\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(\nu + a\tau + b, \tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+a)^2}\, e^{2\pi i (n+a)(\nu+b)} \\ \theta\begin{bmatrix} a+r \\ b+s \end{bmatrix}(\nu,\tau) &= e^{i\pi i s a}\theta\begin{bmatrix} a \\ b \end{bmatrix}(\nu,\tau)\,, \qquad r,s\in\mathbb{Z}\,. \end{aligned} \tag{A.21}$$

Under modular transformations generated by $S:\tau\to-\frac{1}{\tau}$ and $T:\tau\to\tau+1$, we have

$$\begin{aligned} T:\ \theta\begin{bmatrix} a \\ b \end{bmatrix} &\to e^{-i\pi a^2 - i\pi a}\,\theta\begin{bmatrix} a \\ a+b+\frac{1}{2} \end{bmatrix}, \qquad \eta\to e^{i\pi/12}\,\eta \\ S:\ \theta\begin{bmatrix} a \\ b \end{bmatrix} &\to (-i\tau)^{\frac{1}{2}} e^{2\pi i a b}\,\theta\begin{bmatrix} -b \\ a \end{bmatrix}, \qquad \eta\to(-i\tau)^{\frac{1}{2}}\eta\,. \end{aligned} \tag{A.22}$$

Thus under an S-transformation, we have

$$\begin{aligned} \frac{\zeta_l^j}{\eta^4} \to e^{2\pi i \sum_i \frac{jlk_i^2}{N^2}} \Bigg\{ &\prod_{i=1}^{3}\theta\begin{bmatrix} -\frac{lk_i}{N} \\ \frac{jk_i}{N} \end{bmatrix}\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix} - e^{-i\pi\sum_i \frac{jk_i}{N} + 2\pi i \sum_i \frac{1}{2}\frac{jk_i}{N}} \prod_{i=1}^{3}\theta\begin{bmatrix} -\frac{lk_i}{N}-\frac{1}{2} \\ \frac{jk_i}{N} \end{bmatrix}\theta\begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix} - \\ &- e^{2\pi i \sum_i \frac{1}{2}\frac{lk_i}{N}} \prod_{i=1}^{3}\theta\begin{bmatrix} -\frac{lk_i}{N} \\ \frac{jk_i}{N}+\frac{1}{2} \end{bmatrix}\theta\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} - \\ &- e^{-i\pi\sum_i \frac{jk_i}{N} + 2\pi i \sum_i \frac{1}{2}(\frac{jk_i}{N}+\frac{lk_i}{N}) + 2\pi i(\frac{3}{4}+\frac{1}{4})} \prod_{i=1}^{3}\theta\begin{bmatrix} -\frac{lk_i}{N}-\frac{1}{2} \\ \frac{jk_i}{N}+\frac{1}{2} \end{bmatrix}\theta\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\Bigg\}. \end{aligned} \tag{A.23}$$

The sum over such terms can be rewritten as the original partition function with $j' = N-l$, $l' = j$ if the phase from the third term above satisfies

$$e^{i\pi \sum_i \frac{lk_i}{N}} = e^{-i\pi \sum_i \frac{j'k_i}{N}} = e^{-i\pi \sum_i \frac{(N-l)k_i}{N}} . \tag{A.24}$$

In other words, we require

$$\sum_i \frac{(N-l)k_i}{N} = -\sum_i \frac{lk_i}{N} + \text{even} \tag{A.25}$$

i.e. $\sum_i k_i =$ even, the condition we have seen before (this condition on the orbifold weights can also be obtained by demanding level-matching). Invariance under the T transformation does not give anything new.

We can now expand the type-II partition function we have here to realize the GSO projection on the twisted states to obtain the projector

$$1 - (-1)^{-i\pi \sum_i [jk_i/N]} \tag{A.26}$$

for the ground states in the sector where $\{\frac{jk_i}{N}\} < \frac{1}{2}$, i.e. the (c_X, c_Y, c_Z) ring. This is a projector onto twisted states with $\sum_i [jk_i/N] = E_j =$ odd, recovering the result from the previous subsection. On the other hand, consider as an example, the sector where $\{\frac{jk_3}{N}\} > \frac{1}{2}$ with $\{\frac{jk_1}{N}\}, \{\frac{jk_2}{N}\} < \frac{1}{2}$. Then we obtain the projector

$$1 - (-1)^{-i\pi (\sum_i [jk_i/N] - 1)} \tag{A.27}$$

for the ground states (which are in the (c_X, c_Y, a_Z) ring), i.e. $\sum_i [jk_i/N] =$ even. The chiral operators X_j are obtained as the excited state with one extra fermion number from ψ_3 which therefore have the GSO projection $\sum_i [jk_i/N] = E_j =$ odd, as before. Likewise if two of $\{\frac{jk_i}{N}\} > \frac{1}{2}$, we have $E_j =$ odd for the ground states so that the X_j, obtained with one extra fermion number in the two sectors, again have $E_j =$ odd and so on. Thus the GSO exponent for the chiral operators X_j is $E_j = \sum_i [jk_i/N] =$ odd.

The above partition function can be recast as the Green-Schwarz partition function using the quartic Riemann identity for theta functions [13] (see [11, 12] for the $\mathbb{C}/\mathbb{Z}_N$ case). It is noteworthy that the partition function of the type-II theory can be obtained by gauging a chiral $(-1)^{F_L}$ $\mathbb{Z}_2$ symmetry in (i.e. as a $\mathbb{Z}_2$ orbifold of) the partition function of the type-0 theory and demanding modular invariance, along the lines of [36, 37]. In this case, this procedure effectively changes the $\mathbb{Z}_N$ orbifold to a $\mathbb{Z}_N \times \mathbb{Z}_2$. For odd N, this is the same as $\mathbb{Z}_{2N}$ and we recover the partition function above and thence the Green-Schwarz partition function. While in principle one could expect interesting generalizations involving discrete torsion in $\mathbb{Z}_N \times \mathbb{Z}_2$ for even N, a careful calculation shows that the possible extra phases can be absorbed via redefinitions of the orbifold weights.

B. $\mathbb{C}^2/\mathbb{Z}_{N(p)}$ toric geometry

We outline here the toric description of $\mathbb{C}^2/\mathbb{Z}_{N(p)}$ discussed in [5] (see also [2]), based on the Hirzebruch-Jung theory of singularity resolution in codimension two. We have uniformized our notations and conventions with our description of $\mathbb{C}^3/\mathbb{Z}_{N(p,q)}$.

A basis for monomials invariant under the orbifold action is $u = x^N$, $v = x^{-p}y$, so that the ring of holomorphic functions on a neighbourhood of the noncompact $\mathbb{C}^2/\mathbb{Z}_N$ singularity is generated by the monomials $u^{m_1}v^{m_2} = x^{Nm_1-pm_2}y^{m_2}$ for integer m_1, m_2, which gives a cone in the $\mathbb{M}$ lattice bounded by $(1,0)$, (p,N). From these, we read off the vertices of the fan in the $\mathbb{N}$ lattice dual to this

$$\alpha_1 = (N,-p)\,, \qquad \alpha_2 = (0,1)\,. \tag{B.1}$$

This uniformizes our notations and conventions with those of [5]: see e.g. figure 2 therein (see also e.g. figure 1.10 of [2]). The Hirzebruch-Jung formulation of minimal resolution in codimension two ensures that the endpoint of condensation of all tachyons in a given orbifold is always flat space. Thus the volume of any subcone arising from a subdivision by a tachyonic blowup mode is equal to one, so that unlike $\mathbb{C}^3/\mathbb{Z}_{N(p,q)}$, terminal singularities do not exist here.

Furthermore there is a nice continued fraction representation of $\frac{N}{p}$ for $\mathbb{C}^2/\mathbb{Z}_{N(p)}$, in terms of integers a_k, following from the Hirzebruch-Jung theory of minimal resolution of these orbifolds which encodes the relations

$$a_j v_j = v_{j-1} + v_{j+1} \tag{B.2}$$

between the vectors $v_0 = (N,-p)$, $v_{r+1} = (0,1)$, v_j, $j = 1,\ldots,r$ in the toric diagram representing the bounding vectors of the fan as well as the r generators of the chiral ring. Then by iterating the relation above, we can see from the toric diagram that v_k obeys

$$A_k v_k = B_k v_0 + v_{k+1}\,, \tag{B.3}$$

where

$$A_k = a_k - \frac{1}{(a_{k-1} - \cdots)}\,, \qquad A_1 = a_1\,, \qquad B_k = [A_{k-1}\cdot A_{k-2}\cdots A_1]^{-1}\,. \tag{B.4}$$

Then, realizing that (1,0) is an interior point for $p < N$, we have for $v_r = (1,0)$,

$$A_r(1,0) = B_r(N,-p) + (0,1) \tag{B.5}$$

so that $A_r = B_r N$, $1 = B_r p$, giving

$$\frac{N}{p} = A_r = [a_r, a_{r-1}\ldots]\,. \tag{B.6}$$

There does not appear to be any generalization of the continued fraction fraction representation for $\mathbb{C}^3/\mathbb{Z}_{N(p,q)}$.

In this case also, one can study subsequent tachyons type-II theories in light of the chiral GSO projection. For a $\mathbb{C}^2/\mathbb{Z}_{N(p)}$ orbifold that is type II, we have $p =$ odd. A twisted state $T_j = (j, -[\frac{jp}{N}]) \equiv (\frac{j}{N}, \{\frac{jp}{N}\})$ is preserved by the GSO if $[\frac{jp}{N}] =$ odd. Consider blowing up the divisor corresponding to T_j. The subcone $C(0; T_j, \alpha_2)$ can be easily seen to be a

$\mathbb{C}^2/\mathbb{Z}_j\ (1, [\frac{jp}{N}])$ orbifold which clearly admits a type-II projection since $1 + [\frac{jp}{N}] =$ even. If $T_k \in C(0; T_j, \alpha_2)$, solving $T_k = aT_j + b\alpha_2$ gives

$$R'_k = (a,b) = \left(\frac{k}{j}, \frac{k}{j}\left[\frac{jp}{N}\right] - \left[\frac{kp}{N}\right]\right). \tag{B.7}$$

The GSO requires $a_i =$ odd satisfying $a_1 + a_2[\frac{jp}{N}] = 0\,(\mathrm{mod} 2j)$, which is solved by $a_1 = [\frac{jp}{N}], a_2 = -1$. Thus the GSO exponent for T_k works out to

$$E = \left[\frac{jp}{N}\right]\frac{k}{j} - \frac{k}{j}\left[\frac{jp}{N}\right] + \left[\frac{kp}{N}\right] = \left[\frac{kp}{N}\right]. \tag{B.8}$$

Similarly, the subcone $C(0; T_j, \alpha_1)$ is seen to be a $\mathbb{C}^2/\mathbb{Z}_{N\{\frac{jp}{N}\}}\ (p, -[\frac{jp}{N}])$ orbifold (assuming nondegeneracy of the Smith normal form vector). This also clearly admits a type-II projection since $p - [\frac{jp}{N}] =$ even. A sub-twisted state $T_k \in C(0; T_j, \alpha_1)$ satisfies $T_k = aT_j + b\alpha_1$ giving

$$R'_k = (a,b) = \left(\frac{\{\frac{kp}{N}\}}{\{\frac{jp}{N}\}}, \ \frac{k}{N} - \frac{j}{N}\frac{\{\frac{kp}{N}\}}{\{\frac{jp}{N}\}}\right). \tag{B.9}$$

A set of $a_i =$ odd satisfying $a_1 p - a_2[\frac{jp}{N}] = 0\ (\mathrm{mod} 2N\{\frac{jp}{N}\})$, is $a_1 = [\frac{jp}{N}], a_2 = p$, giving for the GSO exponent for T_k

$$E = \left[\frac{kp}{N}\right]. \tag{B.10}$$

Thus originally preserved sub-twisted states remain preserved after a preserved twisted state condenses. The R-charges of subsequent tachyons again expectedly renormalize upwards, i.e. $R'_k > R_k$, since a sub-tachyon is closer to a sub-plane than to the original plane of marginal operators. However due to an interesting convexity property of the tachyonic generators (that appear in the Hirzebruch-Jung continued fraction) inside a given cone of a toric digram (see e.g. [7]), it is straightforward to show that $R'_k \leq 1$ always! Thus there are no geometric terminal singularities in codimension two.

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JHEP08(2004)047